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**2019**

A TORSION JACQUET-LANGLANDS CORRESPONDENCE

Frank CALEGARI & Akshay VENKATESH

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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ISSN: 0303-1179 (print) 2492-5926 (electronic)  
ISBN 978-2-85629-903-6  
doi:10.24033/ast.1075

Directeur de la publication: Stéphane Seuret

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*Mathematical Subject Classification (2010).* — 11F75, 11F80, 11F70, 57M27.

*Mots-clés.* — Représentations de Galois, cohomologie des groupes arithmétiques, programme de Langlands.

*Keywords.* — Galois Representations, Cohomology of Arithmetic Groups, Langlands Program.

# A TORSION JACQUET-LANGLANDS CORRESPONDENCE

par Frank CALEGARI & Akshay VENKATESH

*Abstract.* — We prove a numerical form of a Jacquet-Langlands correspondence for torsion classes on arithmetic hyperbolic 3-manifolds.

*Résumé. (Une correspondance de Jacquet-Langlands torsion)* — Nous établissons une forme numérique d'une correspondance de Jacquet-Langlands pour les classes de torsion sur des variétés hyperboliques arithmétiques de dimension 3.



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## ACKNOWLEDGMENTS

The first author (F.C.) would like to thank Matthew Emerton for many conversations regarding possible integral formulations of reciprocity, and to thank Nathan Dunfield; the original calculations suggesting that a torsion Jacquet-Langlands theorem might be true were done during the process of writing our joint paper [17], and the data thus produced proved very useful for suggesting some of the phenomena we have studied in this book. He would also like to thank Kevin Hutchinson for some helpful remarks concerning a theorem of Suslin. F.C. was supported during the preparation of this book by a Sloan fellowship and by the National Science Foundation (CAREER Grant DMS-0846285, NSF Grant DMS-1404620, NSF Grant DMS-1648702, and NSF Grant DMS-1701703).

The second author (A.V.) would like to express his gratitude to Nicolas Bergeron, with whom he wrote the paper [7], for many fruitful discussions and thoughts about the analytic behavior of analytic torsion; to Laurent Clozel, who suggested to him the importance of investigating torsion in cohomology; to Kartik Prasanna, for helpful discussions and references concerning his work; and to Avner Ash for encouraging words. He was supported during the preparing of this book by a Sloan fellowship, by National Science Foundation grants DMS-1065807 and DMS-1401622, and a David and Lucile Packard Foundation fellowship; he gratefully thanks these organizations for their support.

The mathematical debt this book owes to the work of Cheeger [24] and Müller [68] should be clear: the main result about comparison of torsion homology makes essential use of their theorem.

Some of the original ideas of this book were conceived during the conference “Explicit Methods in Number Theory” in Oberwolfach during the summer of 2007. Various parts of the manuscript were written while the authors were resident at the following institutions: New York University, Stanford University, Northwestern University, the Institute for Advanced Study, the University of Chicago, the University of Melbourne, the University of Sydney, and the Brothers K coffee shop; we thank all of these institutions for their hospitality.

We thank Toby Gee, Florian Herzig, Krzysztof Klosin, Kai-Wen Lan, and Romyar Sharifi for helpful comments on a previous version of this manuscript. We thank Jean Raimbault and Gunter Harder for pointing out errors in Chapter 5. We thank