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LOCAL REGULARITY PROPERTIES OF ALMOST- AND QUASIMINIMAL SETS WITH A SLIDING BOUNDARY CONDITION

par Guy DAVID

Abstract. — We study the boundary regularity of almost minimal and quasiminimal sets that satisfy sliding boundary conditions. The competitors of a set E are defined as $F = \varphi_1(E)$, where $\{\varphi_t\}$ is a one parameter family of continuous mappings defined on E, and that preserve a given collection of boundary pieces. We generalize known interior regularity results, and in particular we show that the quasiminimal sets are locally Ahlfors-regular, rectifiable, and some times uniformly rectifiable, that our classes are stable under limits, and that for almost minimal sets the density of Hausdorff measure in balls centered on the boundary is almost nondecreasing.

Résumé. **(Propriétés de régularité locale des ensembles presque- et quasiminimaux avec une condition de frontière glissante)** — On s'intéresse à la régularité jusqu'à la frontière des ensembles presque minimaux et quasiminimaux sous une condition de glissement. Les compétiteurs d'un ensemble E y sont de la forme $F = \varphi_1(E)$, où $\{\varphi_t\}$ est une famille à un paramètre d'applications continues définies sur E, et qui préservent des ensembles frontières donnés à l'avance. On généralise des résultats connus à l'intérieur, et on démontre notamment l'Ahlfors régularité, la rectifiabilité et parfois l'uniforme rectifiabilité locales des ensembles quasiminimaux, la stabilité des classes considérées par limites, et la presque monotonie de la densité des ensembles presque minimaux sur des boules centrées à la frontière.

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CONTENTS

FREQUENTLY USED NOTATION

$B(x,r) = \{y\,;\, |y-x| < r\}$ is the open ball centered at x and with radius $r > 0$.

$\mathscr{H}^d$ is the d-dimensional Hausdorff measure. See **[22]** or **[32]**.

GSAQ = GSAQ(U, M, δ, h) is a class of quasiminimal sets; see Definition 2.3.

$W_t = \{y \in E \cap B\,; \varphi_t(y) \neq y\}$ and $\widehat{W} = \bigcup_{0<t\leq 1} W_t \cup \varphi_t(W_t)$; see (2.1).

$E^* = \{x \in E\,; \mathscr{H}^d(E \cap B(x,r)) > 0$ for every $r > 0\}$ is the core of E; see (3.2).

$d_{x,r}(E,F)$ is almost a normalized Hausdorff distance in $B(x,r)$; see (10.5).

† † delimits a proof or comment that concerns the Lipschitz assumption only.

$W_f = \{x \in \mathbb{R}^n\,;\, f(x) \neq x\}$; see (11.19).

$\widetilde{f}(x) = \psi(\lambda f(x))$ (used in Part IV, in the Lipschitz case); see (11.50), (12.75).

$B_j = B(x_j, t)$, $j \in J_1$, is our first collection of balls (Part IV); see (12.8)–(12.9).

$B_j = B(x_j, r_j)$, $j \in J_2$, is the second collection of balls; see Lemma 13.

$D_j = B(y_j, r_j)$, $j \in J_3$, balls in the image, are used with the $B_{j,x}$; see (14.12)–(14.14).

$B_{j,x}$, $x \in Z(y_j)$, is our third collection of balls; see (14.19) and (14.1).

$h(r)$ is a gauge function that measures almost minimality; see (19.1) and Definition 19.2.

$\mathscr{I}(U,a,b)$, $\mathscr{I}_l(U,a,b)$, and $\mathscr{I}^+(U,a,b)$ are classes of elliptic integrands; see Definition 24.3, Claim 24.91, and (24.96).

PART I

INTRODUCTION AND DEFINITIONS

CHAPTER 1

INTRODUCTION

The main purpose of this paper is to study the boundary regularity properties of minimal, almost minimal, and quasiminimal sets, subject to sliding boundary conditions that we will explain soon.

A long term motivation is to study various types of Plateau problems, but where the objects under scrutiny are a priori just sets (rather than currents or varifolds), and we want to assume as little structure on them as possible. In this respect, the sliding conditions below seem natural to the author, and should be flexible enough to allow for a variety of applications.

Let us give a very simple example of a Plateau problem that we may want to study, and for which we do not have an existence result yet. Let $\Gamma \subset \mathbb{R}^n$ be a smooth closed curve, and let $E_0 \subset \mathbb{R}^n$ be a compact set that contains Γ. For instance, parameterize Γ by the unit circle, extend the parameterization to the closed unit disk, and let E_0 be the image of the disk. Many other examples are possible, but with this one we should not get a trivial problem for which the infimum is zero. Our Plateau problem consists in minimizing $\mathscr{H}^2(E)$ among all sets E that can be written $E = \varphi_1(E_0)$, where $\{\varphi_t\}$, $0 \leq t \leq 1$, is a continuous, one parameter family of continuous mappings from E_0 to $\mathbb{R}^n$, with $\varphi_0(x) = x$ for $x \in E_0$ and $\varphi_t(x) \in \Gamma$ for $0 \leq t \leq 1$ when $x \in E_0 \cap \Gamma$. Thus, along our deformation of E_0 by the φ_t, we allow the points of Γ to move, but only along Γ; this is why we shall use the term "sliding boundary condition".

Minimizers of this problem, if they exist, will be among our simplest examples of minimal sets with a sliding boundary condition. But solutions of other types of Plateau problems (Reifenberg minimizers as in [R1,2], [**19**], or [**21**], or size minimizing currents under the boundary constraint $\partial T = G$, where G denotes the current of integration along Γ, when they exist, also yield minimal sets with a sliding boundary condition. Thus regularity results for sliding minimal sets may be useful for a variety of problems, and we can also hope that they may help with existence results.

Let us first give some definitions, and then discuss these issues a little more. The sets that we want to study are variants of the Almgren minimal, almost minimal, or quasiminimal sets (he said "restricted sets"), as in [**3**], but where we add boundary constraints and are interested in the behavior of these sets near the boundary.

We work in a closed region Ω of $\mathbb{R}^n$, which may also be $\mathbb{R}^n$ itself, and we give ourselves a finite collections of closed sets $L_j \subset \Omega$, $0 \leq j \leq j_{\max}$, that we call boundary pieces. It will make our notation easier to consider Ω as our first boundary piece, i.e., set

$$L_0 = \Omega. \tag{1.1}$$

For the elementary Plateau problem suggested above, for instance, we would work with $L_0 = \Omega = \mathbb{R}^n$ and $L_1 = \Gamma$.

We are also given an integer dimension d, with $0 \leq d \leq n-1$, and we consider closed sets $E \subset \Omega$, whose d-dimensional Hausdorff measure is locally finite, i.e., such that

$$\mathscr{H}^d(E \cap B(x,r)) < +\infty \tag{1.2}$$

for $x \in \Omega$ and $r > 0$. The next definition explains what we mean by a deformation of E that preserves the boundary pieces.

Definition 1.3. — *Let $B = \overline{B}(y,r)$ be a closed ball in $\mathbb{R}^n$. We say that the closed set $F \subset \Omega$ is a* competitor *for E in B, with sliding conditions given by the closed sets L_j, $0 \leq j \leq j_{\max}$, when $F = \varphi_1(E)$ for some one-parameter family of functions φ_t, $0 \leq t \leq 1$, with the following properties:*

$$(t,x) \to \varphi_t(x) \text{ is a continuous mapping from } [0,1] \times E \text{ to } \mathbb{R}^n, \tag{1.4}$$

$$\varphi_t(x) = x \text{ for } t = 0 \text{ and for } x \in E \setminus B, \tag{1.5}$$

$$\varphi_t(x) \in B \text{ for } x \in E \cap B \text{ and } t \in [0,1], \tag{1.6}$$

and, for $0 \leq j \leq j_{\max}$,

$$\varphi_t(x) \in L_j \text{ when } t \in [0,1] \text{ and } x \in E \cap L_j \cap B. \tag{1.7}$$

We also require that

$$\varphi_1 \text{ be Lipschitz,} \tag{1.8}$$

but with no bounds required.

We shall sometimes say “sliding competitor in B” instead of “competitor for E in B, with sliding conditions given by the L_j, $0 \leq j \leq j_{\max}$,” especially when our choice of Ω and the list of L_j are clear from the context.

We shall soon discuss minimality, almost minimality, and quasiminimality relative to this notion of sliding competitors, but since the class of competitors is often the most important part of the definitions, a number of general comments on Definition 1.3 will be helpful.

It is important here that φ_1 is allowed not to be injective. So we are allowed to merge different portions of E, or contract them to a point, or pinch them in some other way. This, together with the fact that we shall not count measure with multiplicity, is why the union of two parallel disks that lie close to each other will not be minimal.

We added the last requirement (1.8) because Almgren put it in his definitions, and because this will not disturb. If we drop it, we get more competitors for E, which means that the almost- and quasiminimality properties are harder to get. Hence the regularity results proved here are also valid in the context where we drop (1.8). On the other hand, (1.8) will often be easy to prove, so it does not bother us much. The author suspects that the reason why Almgren added (1.8) may be the following. Suppose you want to show that the support of a size minimizing current T is a minimal set and, to simplify the discussion, that you are proceeding locally, in the complement of the boundary sets. You are given a deformation $\{\varphi_t\}$ as in Definition 1.3, and of course the simplest way to use it is to show that pushing T by the φ_t, and in particular φ_1, defines an acceptable competitor for T (with the same boundary constraints). The constraint (1.8) just makes it possible to define the pushforward of T by φ_1, so it is convenient. See **[11]** for details on this argument and its extension to the boundary.

In the other direction, J. Harrison and H. Pugh once asked wether requiring φ_1, or even all the φ_t, to be smooth, would lead to the same classes of almost- and quasiminimal sets. The question was raised in the local context with no boundaries, but it also makes sense in the present context. The answer is yes under suitable conditions on the L_j, and if smooth means C^1. For higher regularity, a proof seems to be manageable, but quite ugly, and so we only give a very rough sketch of how we would proceed, using the construction of Part IV. This is discussed in Chapter 27.

We are allowed to take $\Omega = \mathbb{R}^n$, and then (1.7) for $j = 0$ is just empty and if there is no other boundary piece we get a minor variant of Almgren's definition of competitors in $\mathbb{R}^n$. Of course we can still restrict the list of competitors like he did, by requiring that B lie in a fixed open set U, or that its diameter be less than some $\delta > 0$; we shall do this when we discuss our classes of almost- and quasiminimal sets, but let us not worry for the moment.

The main difference with Almgren's definition comes from the sliding boundary constraint (1.7), and this is also why we insist on the fact that φ_1 is the endpoint of a continuous deformation. If we did not require (1.7), and we were given a continuous mapping φ_1 such that $\varphi_1(x) = x$ for $x \in E \setminus B$ and $\varphi_1(x) \in B$ for $x \in B$, we could define the φ_t by $\varphi_t(x) = t\varphi_1(x) + (1-t)x$, and it is easy to check that (1.4)–(1.6) would hold (because B is convex). We could also extend φ_1 to $\mathbb{R}^n$, which fits with the fact that φ_1 is traditionally defined on $\mathbb{R}^n$, not just on E. But in the present situation we want points of the boundary L_j to stay in L_j (hence, (1.7)), and then it seems natural to say that the deformation condition in (1.7) only concerns points of E: we do not want to say that the air besides our soap film E is also concerned by the sliding boundary constraint. Notice that the φ_t can be extended to $\mathbb{R}^n$ (but in a way that may not preserve the L_j), so we do not have to worry about the case where our deformations would yield a tearing apart (cavitation) of the air besides the soap film.

Notice that with our convention that $L_0 = \Omega$, the set $\varphi_t(E)$ stays in Ω, i.e.,

$$\varphi_t(x) \in \Omega \text{ for } x \in E \text{ and } t \in [0,1], \tag{1.9}$$

either because $x \in E \setminus B$ and $\varphi_t(x) = x \in E \subset \Omega$ by (1.5), or else by (1.1) and (1.7) with $j = 0$.

The author thinks that Definition 1.3 is a nice way to encode boundary constraints, for instance that would be satisfied when E is a soap film in a domain. A Plateau boundary constraint could for instance be associated to one or a few curves L_j, but we could also think about $L_1 = \partial\Omega$ (or some other surface) as being a boundary along which the soap film may slide (as if loosely attached to a wall). It is quite probable that such boundary conditions were studied in the past, but the author does not know where. In **[38]**, J. Taylor came close to giving a similar definition in a slightly different context (flat chains modulo 2), but apparently forgot to state (1.7) or another similar condition.

Once we have a notion of competitors, we can define a corresponding notion of minimal sets. Let us say, for the moment, that the closed set $E \subset \Omega$ is minimal, with the sliding boundary conditions defined by the L_j, $0 \leq j \leq j_{\max}$, if $H^d(E) < +\infty$ and

$$(1.10) \quad \mathscr{H}^d(E) \leq \mathscr{H}^d(F) \text{ whenever } F \text{ is a sliding competitor for } E \text{ in some ball } B,$$

where we allow B to depend on F. Many variants of this definition will be proposed, where one may localize the definition to an open set U, or add a small error term to the right-hand side in (1.10) (this is how we will define almost minimal sets), or even allow stronger distortions (this will give rise to quasiminimal sets). We shall give the main definitions in Chapter 2 (for the generalized quasiminimal sets) and later in Chapter 20 (for almost minimal sets), but for the moment the sliding minimal sets that satisfy (1.10) will give a fair idea of what we want to study.

Of course our notion of competitors can be used to define Plateau problems, as we did earlier with a single curve. Given a collection of boundary pieces L_j, and a closed set E_0, we can try to minimize $\mathscr{H}^d(E)$ among all the sets E that are sliding competitors of E_0 (in some ball B that depends on E, or in some fixed huge ball that contains Ω). If E_0 is badly chosen (for instance, if some sliding competitors of E_0 are reduced to a point), the problem may not be interesting, but it is easy to produce lots of examples where the infimum will be finite and positive. For most of these examples, we do not have an existence result. But it is clear that if minimizers for this Plateau problem exist, they are sliding minimal sets.

The main point of this paper is to study the general (hence often rather weak) regularity properties of the minimal sets, and their almost minimal and quasiminimal variants, in particular when we approach the boundary pieces L_j. In practical terms, this means that we will take many interior regularity results for Almgren minimal (or quasiminimal) sets, and try to adapt their proofs so that they work all the way to the boundary. But before we say more about this, let us comment a little more on Definition 1.3 and our motivations.

The word sliding may be misleading in some cases, as some sets L_j may be reduced to points, where in effect no sliding will be allowed. Our assumptions on the L_j will only allow a finite number of points where E is fixed. So, for instance, we do not consider the case where Γ is a simple curve and we require that $\varphi_t(x) = x$ for every point $x \in E \cap \Gamma$. This will not bother us, and probably such a condition would make it too hard to produce competitors and get information on E near Γ when E is a minimal set with these constraints. Of course we could always say that E is locally minimal (for instance) in the domain $U = \mathbb{R}^n \setminus \Gamma$, and get some information from this, but this is not the point of this paper. On the contrary, the author believes that because we allow our competitors to slide along the L_j, we will have an amount of flexibility in the construction of competitors, which we can use to prove some decent regularity results. And at the same time (1.7) looks like a reasonable constraint, for instance, if we want to model the behavior of soap films.

We believe that in addition to being interesting by themselves, regularity results for sliding minimal or almost sets could be useful to prove existence results (in very simple cases) for the Plateau problems discussed above, and also for other similar problems, because some other types of minimizers also yield sliding minimal sets. Let us give two examples.

In **[35]**, Reifenberg proposed a Plateau problem where we are given a compact boundary set $L \subset \mathbb{R}^n$ of dimension $d-1$, and we minimize $\mathscr{H}^d(E)$ among compact sets E that bound L, in the sense that $L \subset E$ and the natural map induced by the inclusion, from the $(d-1)$-dimensional Čech homology group of L to the $(d-1)$-dimensional Čech homology group of E, is trivial. He also proves a fairly general existence result, and good interior regularity results for the minimizers (see [R1,2]). These results were generalized by various authors; see for instance **[1]**, **[19]**, and more recently **[21]** for a quite general existence result. Also see **[26]** for a simpler variant of **[35]** in codimension 1, where one replaces the computation of Čech homology groups with a simpler linking condition, and which comes with a simpler proof and is related to differential chains.

It is easy to see that if the boundary set L is not too ugly, the minimizing sets that are obtained in these papers are sliding minimal sets associated to $L_0 = \mathbb{R}^n$ and $L_1 = L$. See **[11]** for the rather easy verification, whose main point is just that if E bounds L and F is a sliding competitor for E, then F bounds L too.

Reifenberg's homological Plateau problem and its minimizers are very nice, and give good descriptions of many soap films, but some people prefer the related problem of size minimizers. That is, we are given a $(d-1)$-dimensional integral current S, with $\partial S = 0$, and we look for a d-dimensional integral current T such that $\partial T = S$ and whose size (understand, the H^d-measure of the set where the multiplicity is nonzero, but we shall be slightly sloppy on the definitions) is minimal. If $d = 2$, L is a nice closed curve in $\mathbb{R}^3$, and S is the current of integration on L, T. De Pauw showed in **[19]** that the infimum for this problem is the same as for Reifenberg's homological problem (where Čech homology is computed over the group $\mathbb{Z}$); but even though De Pauw showed that Reifenberg homological minimizers exist, size minimizers are not known yet to exist. Anyway, size minimizers, if they exist, are also supported (under reasonable conditions) on sliding minimal sets. The point now is that if T is supported by the closed set E and F is a sliding competitor for E, then we can use φ_1 to push T