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A LOCAL TRACE FORMULA FOR THE GAN-GROSS-PRASAD CONJECTURE FOR UNITARY GROUPS: THE ARCHIMEDEAN CASE

by Raphaël BEUZART-PLESSIS

Abstract. — In this volume, we prove, inspired by earlier work of Waldspurger on orthogonal groups, a sort of local trace formula which is related to the local Gan-Gross-Prasad conjecture over any local field F of characteristic zero. As a consequence, we obtain a geometric formula for certain multplicities $m(\pi)$ appearing in this conjecture and deduce from it a weak form of the local Gan-Gross-Prasad conjecture (multiplicity one in tempered L-packets). These results were already known over p-adic fields by previous work of the author and thus are only new when $F = \mathbb{R}$. However, the proof we present here works uniformly over all local fields of characteristic zero.

Résumé. **(Une formule de traces locale reliée à la conjecture de Gan-Gross-Prasad pour les groupes unitaires)** — Dans cet ouvrage, on établit, en s'inspirant de travaux antérieurs de Waldspurger pour les groupes orthogonaux, une sorte de formule des traces relative reliée à la conjecture locale de Gan-Gross-Prasad pour les groupes unitaires sur un corps local F de caractéristique nulle. Comme conséquence, on obtient une formule géométrique pour certaines multiplicités $m(\pi)$ apparaissant dans cette conjecture dont on déduit une forme faible de la conjecture locale de Gan-Gross-Prasad (multiplicité un dans les L-paquets tempérés). Ces résultats étaient déjà connus pour les corps p-adiques, d'après un travail précédent de l'auteur, et ne sont donc nouveaux que pour $F = \mathbb{R}$. Cependant, la preuve présentée ici s'applique uniformément à tous les corps locaux de caractéristique zéro.

CONTENTS

INTRODUCTION

Let F be a local field of characteristic 0 which is different from $\mathbb{C}$. So, F is either a p-adic field (that is a finite extension of $\mathbb{Q}_p$) or $F = \mathbb{R}$. Let E/F be a quadratic extension of F (if $F = \mathbb{R}$, we have $E = \mathbb{C}$) and let $W \subset V$ be a pair of Hermitian spaces having the following property: the orthogonal complement $W^\perp$ of W in V is odd-dimensional and its unitary group $U(W^\perp)$ is quasi-split. To such a pair (that we call an admissible pair, cf. Section 6.2), Gan, Gross and Prasad associate a triple (G, H, ξ). Here, G is equal to the product $U(W) \times U(V)$ of the unitary groups of W and V, H is a certain algebraic subgroup of G and $\xi : H(F) \to \mathbb{S}^1$ is a continuous unitary character of the F-points of H. In the case where $\dim(W^\perp) = 1$, we just have $H = U(W)$ embedded in G diagonally and the character ξ is trivial. For the definition in codimension greater than 1, we refer the reader to Section 6.2. We call a triple like (G, H, ξ) (constructed from an admissible pair (W, V)) a GGP triple.

Let π be a tempered irreducible representation of $G(F)$. By this, we mean that π is an irreducible unitary representation of $G(F)$ whose coefficients satisfy a certain growth condition (an equivalent condition is that π belongs weakly to the regular representation of $G(F)$). We denote by π^∞ the subspace of smooth vectors in π. This subspace is $G(F)$-invariant and carries a natural topology (if $F = \mathbb{R}$, this topology makes π^∞ into a Fréchet space whereas if F is p-adic the topology on π^∞ doesn't play any role but in order to get a uniform treatment we endow π^∞ with its finest locally convex topology). Following Gan, Gross and Prasad, we define a multiplicity $m(\pi)$ by

$$m(\pi) = \dim \operatorname{Hom}_H(\pi^\infty, \xi),$$

where $\operatorname{Hom}_H(\pi^\infty, \xi)$ denotes the space of continuous linear forms ℓ on π^∞ satisfying the relation $\ell \circ \pi(h) = \xi(h)\ell$ for all $h \in H(F)$. By the main result of **[33]** (in the real case) and **[1]** (in the p-adic case) together with Theorem 15.1 of **[26]**, we know that this multiplicity is always less or equal to 1.

The main result of this paper extends this multiplicity one result to a whole L-packet of tempered representations of $G(F)$. This answers a conjecture of Gan, Gross and Prasad (Conjecture 17.1 of **[26]**). Actually, the result is better stated if we consider more than one GGP triple at the same time. In any family of GGP triples that we are going to consider there is a distinguished one corresponding to the case where G and H are quasi-split over F. So, for convenience, we assume that the GGP triple (G, H, ξ) we started with satisfies this condition. The other GGP triples that we need to consider may be called the pure inner forms of (G, H, ξ). Those are naturally parametrized by the Galois cohomology set $H^1(F, H)$. A cohomology class $\alpha \in H^1(F, H)$ corresponds to a Hermitian space W_α (up to isomorphism) of the same dimension as W. If we set $V_\alpha = W_\alpha \oplus^\perp W^\perp$, then (W_α, V_α) is an admissible pair and thus gives rise to a new GGP triple $(G_\alpha, H_\alpha, \xi_\alpha)$. The pure inner forms of (G, H, ξ) are exactly all the GGP triples obtained in this way.

Let φ be a tempered Langlands parameter for G. According to the local Langlands correspondence (which is now known in all cases for unitary groups, cf. [**34**] and [**48**]), this parameter determines an L-packet $\Pi^G(\varphi)$ consisting of a finite number of tempered representations of $G(F)$. Actually, this parameter also defines L-packets $\Pi^{G_\alpha}(\varphi)$ of tempered representations of $G_\alpha(F)$ for all $\alpha \in H^1(F, H)$. We can now state the main result of this paper as follows (cf. Theorem 12.4.1).

Theorem 1. — *There exists exactly one representation π in the disjoint union of L-packets*

$$\bigsqcup_{\alpha \in H^1(F,H)} \Pi^{G_\alpha}(\varphi)$$

such that $m(\pi) = 1$.

As we said, this answers in the affirmative a conjecture of Gan-Goss-Prasad (Conjecture 17.1 of [**26**]). The analog of this theorem for special orthogonal groups has already been obtained by Waldspurger in the case where F is p-adic [**59**]. In [**17**], the author adapted the proof of Waldspurger to deal with unitary groups but again under the assumption that F is p-adic. Hence, the only new result contained in Theorem 1 is when $F = \mathbb{R}$. However, the proof we present here differs slightly from the original treatment of Waldspurger and we feel that this new approach is more amenable to generalizations. This is the main reason why we are including the p-adic case in this paper. Actually, it doesn't cost much: in many places, we have been able to treat the two cases uniformly and when we needed to make a distinction, it is often because the real case is more tricky.

As in [**59**] and subsequently [**17**], Theorem 1 follows from a formula for the multiplicity $m(\pi)$. This formula express $m(\pi)$ in terms of the Harish-Chandra character of π. Recall that, according to Harish-Chandra, there exists a smooth function θ_π on the regular locus $G_{\text{reg}}(F)$ of $G(F)$ which is locally integrable on $G(F)$ and such that

$$\text{Trace } \pi(f) = \int_{G(F)} \theta_\pi(x) f(x) dx$$

for all $f \in C_c^\infty(G(F))$ (here $C_c^\infty(G(F))$ denotes the space of smooth and compactly supported functions on $G(F)$). This function θ_π is obviously unique and is called the Harish-Chandra character of π. To state the formula for the multiplicity, we need to extend the character θ_π to a function

$$c_\pi : G_{\text{ss}}(F) \to \mathbb{C}$$

on the semi-simple locus $G_{ss}(F)$ of $G(F)$. If $x \in G_{reg}(F)$, then $c_\pi(x) = \theta_\pi(x)$ but for a general element $x \in G_{ss}(F)$, $c_\pi(x)$ is in some sense the main coefficient of a certain local expansion of θ_π near x. For a precise definition of the function c_π, we refer the reader to Section 4.5, where we consider more general functions that we call quasi-characters and which are smooth functions on $G_{reg}(F)$ sharing almost all of the good properties that characters of representations have. As we said, it is through the function c_π that the character θ_π will appear in the multiplicity formula. The other main ingredient of this formula is a certain space $\Gamma(G, H)$ of semi-simple conjugacy classes in $G(F)$. For a precise definition of $\Gamma(G, H)$, we refer the reader to Section 11.2. Let us just say that $\Gamma(G, H)$ comes naturally equipped with a measure dx on it and that this measure is not generally supported in the regular locus. For example, the trivial conjugacy class $\{1\}$ is an atom for this measure whose mass is equal to 1. Apart from these two main ingredients (the function c_π and the space $\Gamma(G, H)$), the formula for the multiplicity involves two normalizing functions D^G and Δ. Here, D^G is the usual discriminant whereas Δ is some determinant function that is defined in Section 11.2. We can now state the formula for the multiplicity as follows (cf. Theorem 11.4.2).

Theorem 2. — *For every irreducible tempered representation π of $G(F)$, we have the equality*

$$m(\pi) = \lim_{s \to 0^+} \int_{\Gamma(G,H)} c_\pi(x) D^G(x)^{1/2} \Delta(x)^{s-1/2} dx.$$

The integral in the right hand side of the equality above is absolutely convergent for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$ and moreover the limit as $s \to 0^+$ exists (cf. Proposition 11.2.1).

As we said, Theorem 1 follows from Theorem 2. This is proved in the last chapter of this paper (Chapter 12). Let us fix a tempered Langlands parameter φ for G. The main idea of the proof, the same as for Theorem 13.3 of [**59**], is to show that the sum

$$\sum_{\alpha \in H^1(F,H)} \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi), \tag{0.0.1}$$

when expressed geometrically through Theorem 2 contains a lot of cancelations which roughly come from the character relations between the various stable characters associated to φ on the pure inner forms of G. Once these cancelations are taken into account, the only remaining term is the term corresponding to the conjugacy class of the identity inside $\Gamma(G, H)$. By classical results of Rodier and Matumoto, this last term is related to the number of generic representations inside the quasi-split L-packet $\Pi^G(\varphi)$. By the generic packet conjecture, which is now known for unitary groups, we are able to show that this term is equal to 1 and this immediately implies Theorem 1. Let us now explain in more detail how it works. Fix momentarily $\alpha \in H^1(F, H)$. Using Theorem 2, we can express the sum

$$\sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi)$$

as

$$\lim_{s\to 0^+}\int_{\Gamma(G_\alpha,H_\alpha)} c_{\varphi,\alpha}(x)D^{G_\alpha}(x)^{1/2}\Delta(x)^{s-1/2}dx, \tag{0.0.2}$$

where we have set $c_{\varphi,\alpha}=\sum_{\pi\in\Pi^{G_\alpha}(\varphi)}c_\pi$. One of the main properties of L-packets is that the sum of characters $\theta_{\varphi,\alpha}=\sum_{\pi\in\Pi^{G_\alpha}(\varphi)}\theta_\pi$ defines a function on $G_{\alpha,\mathrm{reg}}(F)$ which is stable, which here means that it is invariant by $G_\alpha(\overline{F})$-conjugation. In Section 12.1, we define a notion of strongly stable conjugacy for semi-simple elements of $G_\alpha(F)$. This definition of stable conjugacy differs from the usually accepted one (cf. [**40**]) and is actually stronger (hence the use of the word "strongly"). The point of introducing such a notion is the following: it easily follows from the stability of $\theta_{\varphi,\alpha}$ that the function $c_{\varphi,\alpha}$ is constant on semi-simple strongly stable conjugacy classes. This allows us to further transform the expression 0.0.2 to write it as

$$\lim_{s\to 0^+}\int_{\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)} |p^{-1}_{\alpha,\mathrm{stab}}(x)|c_{\varphi,\alpha}(x)D^{G_\alpha}(x)^{1/2}\Delta(x)^{s-1/2}dx,$$

where $\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$ denotes the space of strongly stable conjugacy classes in $\Gamma(G_\alpha,H_\alpha)$ and $p_{\alpha,\mathrm{stab}}$ stands for the natural projection $\Gamma(G_\alpha,H_\alpha)\twoheadrightarrow\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$ (thus $|p^{-1}_{\alpha,\mathrm{stab}}(x)|$ is just the number of conjugacy classes in $\Gamma(G_\alpha,H_\alpha)$ belonging to the strongly stable conjugacy class of x). Returning to the sum 0.0.1, we can now write it as

$$\sum_{\alpha\in H^1(F,H)}\lim_{s\to 0^+}\int_{\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)} |p^{-1}_{\alpha,\mathrm{stab}}(x)|c_{\varphi,\alpha}(x)D^{G_\alpha}(x)^{1/2}\Delta(x)^{s-1/2}dx. \tag{0.0.3}$$

A second very important property of L-packets is that the stable character $\theta_{\varphi,\alpha}$ is related in a simple manner to the stable character $\theta_{\varphi,1}$ on the quasi-split form $G(F)$. More precisely, Kottwitz [**41**] has defined a sign $e(G_\alpha)$ such that we have $\theta_{\varphi,\alpha}(y)=e(G_\alpha)\theta_{\varphi,1}(x)$ as soon as $y\in G_{\alpha,\mathrm{reg}}(F)$ and $x\in G_{\mathrm{reg}}(F)$ are stably conjugate regular elements (i.e., are conjugate over the algebraic closure where $G_\alpha(\overline{F})=G(\overline{F})$). Once again, this relation extends to the functions $c_{\varphi,\alpha}$ and $c_{\varphi,1}$ and we have $c_{\varphi,\alpha}(y)=e(G_\alpha)c_{\varphi,1}(x)$ for all strongly stably conjugate elements $y\in G_{\alpha,\mathrm{ss}}(F)$ and $x\in G_{\mathrm{ss}}(F)$. As it happens, and contrary to the regular case, there might exist semi-simple elements in $G_\alpha(F)$ which are not strongly stably conjugate to any element of the quasi-split form $G(F)$. However, we can show that the function $c_{\varphi,\alpha}$ vanishes on such elements $x\in G_{\alpha,\mathrm{ss}}(F)$. Therefore, these conjugacy classes don't contribute to the sum 0.0.3 and transferring the remaining terms to $G(F)$, we can express 0.0.3 as a single integral

$$\lim_{s\to 0^+}\int_{\Gamma(G,H)}\left(\sum_{y\sim_{\mathrm{stab}}x} e(G_{\alpha(y)})\right)c_{\varphi,1}(x)D^G(x)^{1/2}\Delta(x)^{s-1/2}dx,$$

where the sum

$$\sum_{y\sim_{\mathrm{stab}}x} e(G_{\alpha(y)}) \tag{0.0.4}$$

is over the conjugacy classes y in the disjoint union $\bigsqcup_{\alpha\in H^1(F,H)}\Gamma(G_\alpha,H_\alpha)$ that are strongly stably conjugate to x and $\alpha(y)\in H^1(F,H)$ denotes the only cohomology class such that y lives in $\Gamma(G_{\alpha(y)},H_{\alpha(y)})$. There is a natural anisotropic torus $T_x\subset H$ associated to $x\in\Gamma_{\mathrm{stab}}(G,H)$ such that the set of conjugacy classes in $\bigsqcup_{\alpha\in H^1(F,H)}\Gamma(G_\alpha,H_\alpha)$ lying inside the strongly stable conjugacy class of x is naturally in bijection with $H^1(F,T_x)$ (cf. Section 12.5 for the definition of T_x). Moreover, for $y\in H^1(F,T_x)$, the cohomology class $\alpha(y)$ is just the image of y via the natural map $H^1(F,T_x)\to H^1(F,H)$. Hence, the sum 0.0.4 equals

$$\sum_{y\in H^1(F,T_x)} e(G_{\alpha(y)}). \tag{0.0.5}$$

In order to further analyze this sum, we need to recall the definition of the sign $e(G_\alpha)$. In **[41]**, Kottwitz constructs a natural map $H^1(F,G)\to H^2(F,\{\pm1\})=Br_2(F)$ from $H^1(F,G)$ to the 2-torsion subgroup of the Brauer group of F. Since F is either p-adic or real, we have an isomorphism $Br_2(F)\simeq\{\pm1\}$. The sign $e(G_\alpha)$ for $\alpha\in H^1(F,H)$ is now just the image of α by the composition of this map with $H^1(F,H)\to H^1(F,G)$. Following Kottwitz's definition, it is not hard to see that the composition $H^1(F,T_x)\to H^1(F,G)\to Br_2(F)$ is a group homomorphism. Moreover, it turns out that for $x\neq 1$ this homomorphism is surjective and this immediately implies that for such an x the sum 0.0.5 is zero. Going back to 0.0.3, we are only left with the contribution of $1\in\Gamma(G,H)$ which is equal to

$$c_{\varphi,1}(1).$$

By a result of Rodier **[49]** in the p-adic case and of Matumoto **[46]** in the real case, the term $c_{\varphi,1}(1)$ has an easy interpretation in terms of Whittaker models. More precisely, this term equals the number of representations in the L-packet $\Pi^G(\varphi)$ having a Whittaker model, a representation being counted as many times as the number of types of Whittaker models it has, divided by the number of types of Whittaker models for $G(F)$. A third important property of L-packets is that $\Pi^G(\varphi)$ contains exactly one representation having a Whittaker model of a given type. It easily follows from this that $c_{\varphi,1}(1)=1$. Hence, the sum 0.0.1 equals 1 and this ends our explanation of how Theorem 2 implies Theorem 1.

The proof of Theorem 2 is more involved and takes up most of this paper. It is at this point that our strategy differs from the one of Waldspurger. In what follows, we explain the motivations and the main steps of the proof of Theorem 2. Consider the unitary representation $L^2(H(F)\backslash G(F),\xi)$ of $G(F)$. It is the L^2-induction of the character ξ from $H(F)$ to $G(F)$ and it consists in the measurable functions $\varphi\colon G(F)\to\mathbb{C}$ satisfying the relation $\varphi(hg)=\xi(h)\varphi(g)$ $(h\in H(F),\ g\in G(F))$ almost everywhere and such that

$$\int_{H(F)\backslash G(F)}|\varphi(x)|^2dx<\infty.$$

The action of $G(F)$ on $L^2(H(F)\backslash G(F),\xi)$ is given by right translation. Since the triple (G,H,ξ) is of a very particular form, the direct integral decomposition