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THE YANG-MILLS HEAT FLOW AND THE CALORIC GAUGE

by Sung-Jin OH & Daniel TATARU

Abstract. — This is the first part of the four-paper sequence, which establishes the Threshold Conjecture and the Soliton Bubbling vs. Scattering Dichotomy for the energy critical hyperbolic Yang-Mills equation in the $(4 + 1)$-dimensional Minkowski space-time.

The primary subject of this paper, however, is another PDE, namely the energy critical Yang-Mills heat flow on the 4-dimensional Euclidean space. Our first goal is to establish sharp criteria for global existence and asymptotic convergence to a flat connection for this system in $\dot{H}^1$, including the Dichotomy Theorem (i.e., either the above properties hold or a harmonic Yang-Mills connection bubbles off) and the Threshold Theorem (i.e., if the initial energy is less than twice that of the ground state, then the above properties hold). Our second goal is to use the Yang-Mills heat flow in order to define the caloric gauge, which will play a major role in the analysis of the hyperbolic Yang-Mills equation in the subsequent papers.

Résumé. **(L'équation de la chaleur de Yang-Mills et la jauge calorique)** — Cet article est la première partie d'une série de quatre articles, qui établit la conjecture de seuil et la dichotomie Soliton Bubbling vs. Scattering pour l'équation hyperbolique de Yang-Mills à énergie critique dans l'espace-temps de Minkowski à $(4 + 1)$ dimensions.

Cependant, le sujet principal de cet article est une autre EDP, à savoir l'equation de la chaleur de Yang-Mills a énergie critique sur l'espace Euclidien à 4 dimensions. Notre premier objectif est d'établir des critères précis concernant l'existence et la convergence asymptotique vers une connexion plate pour ce système dans $\dot{H}^1$, y compris le théorème de dichotomie (c'est-à-dire, soit les propriétés ci-dessus sont valables, soit une connexion harmonique Yang-Mills bubbles off) et le théorème du seuil (c'est-à-dire si l'énergie initiale est inférieure á deux fois celle de l'état fondamental, alors les propriétés ci-dessus sont vraies). Notre deuxième objectif est d'utiliser l'equation de la chaleur de Yang-Mills afin de définir la jauge calorique, qui jouera un rôle majeur dans l'analyse de l'équation hyperbolique de Yang-Mills dans les articles suivants.

CONTENTS

CHAPTER 1

INTRODUCTION

The goal of this paper is two-fold:

— *To develop a large data global theory of the Yang-Mills heat flow on* $\mathbb{R}^4$*.* Consider the Yang-Mills heat flow on $\mathbb{R}^4$ with a compact structure group. For initial data $a \in \dot{H}^1(\mathbb{R}^4)$, we establish sharp criteria for global existence and asymptotic convergence to a flat connection, including the Dichotomy Theorem (Theorem 2.10) and the Threshold Theorem (Theorem 2.11).

— *To define the caloric gauge for the hyperbolic Yang-Mills equation.* Using the large data global theory of the Yang-Mills heat flow, we define the so-called *caloric gauge* (Definition 2.17) and identify the structure of the hyperbolic Yang-Mills equation in this gauge (Theorem 2.27).

While this paper is primarily devoted to the analysis of the Yang-Mills heat flow, in the larger scheme of things it constitutes the first part of a four-paper sequence, whose overall aim is to prove the Threshold Conjecture and the Dichotomy Theorem for the energy critical *hyperbolic* Yang-Mills equation in $\mathbb{R}^{1+4}$. The four installments of the series are concerned with the following topics:

1. the *caloric gauge* for the hyperbolic Yang-Mills equation, present paper.
2. large data *energy dispersed* caloric gauge solutions, **[21]**.
3. *topological classes* of connections and large data local well-posedness, **[19]**.
4. *soliton bubbling* vs. scattering dichotomy for large data solutions, **[22]**.

A short overview of the whole sequence is provided in the survey paper **[20]**.

In the remainder of the introduction, we formulate the three Yang-Mills equations that play a role in this paper, namely the harmonic Yang-Mills equation (elliptic), the Yang-Mills heat flow (parabolic) and the hyperbolic Yang-Mills equation. Then in Section 2, the main results are stated in a more precise form, along with discussion of some major ideas.

1.1. Lie groups and Lie algebras

Let $\mathbf{G}$ be a compact noncommutative Lie group and $\mathfrak{g}$ its associated Lie algebra. We denote by $\mathrm{Ad}(O)X = OXO^{-1}$ the action of $\mathbf{G}$ on $\mathfrak{g}$ by conjugation (i.e., the adjoint action), and by $\mathrm{ad}(X)Y = [X,Y]$ the associated action of $\mathfrak{g}$, which is given by the Lie bracket. We introduce the notation $\langle X, Y\rangle$ for a bi-invariant inner product on $\mathfrak{g}$,

$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle, \qquad X,Y,Z \in \mathfrak{g},$$

or equivalently

$$\langle X, Y\rangle = \langle \mathrm{Ad}(O)X, \mathrm{Ad}(O)Y\rangle, \qquad X, Y \in \mathfrak{g}, \quad O \in \mathbf{G}.$$

If $\mathbf{G}$ is semisimple then one can take $\langle X, Y\rangle = -\mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ i.e., the negative of the Killing form on $\mathfrak{g}$, which is then positive definite. However, a bi-invariant inner product on $\mathfrak{g}$ exists for any compact Lie group $\mathbf{G}$.

1.2. Connections and curvature

The objects of study here are connection 1-forms taking values in the Lie algebra $\mathfrak{g}$,

$$A_j : \mathbb{R}^d \to \mathfrak{g}.$$

The associated covariant differentiation operators $\mathbf{D}_j = (\mathbf{D}^{(A)})_j$, acting on $\mathfrak{g}$-valued functions B, are defined by

$$\mathbf{D}_j B := \partial_j B + \mathrm{ad}(A_j)B.$$

Their commutators yield the curvature tensor

$$F_{jk} := \partial_j A_k - \partial_k A_j + [A_j, A_k],$$

in the sense that $\mathbf{D}_j\mathbf{D}_k - \mathbf{D}_k\mathbf{D}_j = \mathrm{ad}(F_{jk})$. A basic property of F is that it satisfies the *Bianchi identity*:

$$\mathbf{D}_\alpha F_{\beta\gamma} + \mathbf{D}_\beta F_{\gamma\alpha} + \mathbf{D}_\gamma F_{\alpha\beta} = 0.$$

Given a $\mathbf{G}$-valued function O, its action $B \to \mathrm{Ad}(O)B$ induces a gauge transformation for A, namely

$$A_k \to OA_kO^{-1} - \partial_k O O^{-1} =: \mathcal{G}(O)A_k.$$

Correspondingly, we have for F

$$F_{jk} \to OF_{jk}O^{-1}.$$

1.3. Yang-Mills equations

While this article is primarily devoted to the Yang-Mills heat flow, there are in effect three Yang-Mills equations which play a role in our work. These are as follows:

1.3.1. The harmonic Yang-Mills equations in the Euclidean space $\mathbb{R}^d$. — This is obtained as the Euler-Lagrange equation for the Lagrangian (or the energy)

$$\mathcal{E}_e[A] := \frac{1}{2}\int_{\mathbb{R}^d} \langle F_{jk}, F^{jk}\rangle \, dx, \tag{1.1}$$

and has the form

$$\mathbf{D}^j F_{jk} = 0. \tag{1.2}$$

Both the Lagrangian and the harmonic Yang-Mills equation are invariant with respect to gauge transformations, therefore in order to have a good theory for these equations one needs to fix the gauge. A common choice here is the *Coulomb gauge*,

$$\partial^j A_j = 0, \tag{1.3}$$

which formally turns the Equations (1.2) into a strongly elliptic system,

$$\Delta_A A_k = -[A^j, \mathbf{D}_k A_j], \tag{1.4}$$

where Δ_A is the covariant Laplacian, given by

$$\Delta_A = \mathbf{D}^j \mathbf{D}_j. \tag{1.5}$$

1.3.2. The Yang-Mills heat flow in $\mathbb{R}^+ \times \mathbb{R}^d$. — This can be viewed as the gradient flow for the above Lagrangian, but written in a gauge invariant fashion. Using the letter s for the heat-time, we add a heat-time connection component A_s and the corresponding curvatures F_{sj}. Then the covariant Yang-Mills heat flow has the form

$$F_{sj} = \mathbf{D}^\ell F_{\ell j}, \qquad A_j(s=0) = a_j. \tag{1.6}$$

The solutions of the harmonic Yang-Mills equation play the role of steady states for the Yang-Mills heat flow.

This flow is also gauge invariant. A natural way to fix the gauge is via the *de Turck gauge condition*

$$A_s = \partial^k A_k, \tag{1.7}$$

which formally turns the system (1.6) into a semilinear strongly parabolic system,

$$(\partial_s - \Delta_A) A_j = [A_j, \partial^k A_k] - [A^k, \partial_j A_k]. \tag{1.8}$$

However, there is a second gauge choice which plays the leading role in this article, namely the *local caloric gauge* (1),

$$A_s = 0. \tag{1.9}$$

In this gauge the system (1.6) becomes a semilinear degenerate parabolic system,

$$(\partial_s - \Delta_A) A_j = -\mathbf{D}^k \partial_j A_k, \tag{1.10}$$

where the degenerate part occurs at the level of the divergence of A, namely

$$\partial_s(\partial^k A_k) = [A^j, \mathbf{D}^k F_{jk}]. \tag{1.11}$$

This would seem to be less favorable from an analytic point of view. However, as it turns out, under this gauge the long-time behavior is better. Incidentally, this is exactly the gauge which corresponds to directly taking the gradient flow for the Lagrangian in (1.1).

1. The word *local* is here to differentiate this gauge with the global caloric gauge defined in Definition 2.17 below.

Remark 1.1. — In the literature, the gradient flow $\partial_s A_j = \mathbf{D}^\ell F_{\ell j}$ is usually called the *Yang-Mills heat flow.* In our work, however, we find it conceptually and technically useful to adopt the fully gauge-covariant formulation (1.6), and view this flow as the equation in a gauge defined by (1.9). Of course, these viewpoints are equivalent.

The first part of the paper will be devoted to the study of global solutions for the Yang-Mills heat flow.

1.3.3. The hyperbolic Yang-Mills equation in the Minkowski space $\mathbb{R}^{1+d}$. — Let $\mathbb{R}^{1+d}$ be the $(d+1)$-dimensional Minkowski space, equipped with the Minkowski metric $\mathrm{diag}(-1,+1,\dots,+1)$ in the rectangular coordinates $(x^0, x^1, \dots, x^d)$.

The *hyperbolic Yang-Mills equations* are the Euler-Lagrange equations associated with the formal Lagrangian action functional

$$\mathscr{L}[A] := \frac{1}{2}\int_{\mathbb{R}^{1+d}} \langle F_{\alpha\beta}, F^{\alpha\beta}\rangle \, dxdt. \tag{1.12}$$

Here we are using the standard convention for raising indices using the Minkowski metric, and greek letters for the Minkowski setting. The (hyperbolic) time is denoted by t, and corresponds to the index 0 (i.e., $t = x^0$). Thus, the hyperbolic Yang-Mills equations take the form

$$\mathbf{D}^\alpha F_{\alpha\beta} = 0. \tag{1.13}$$

In order to consider the Yang-Mills problem as an evolution equation we need to consider initial data sets. An *initial data set* for (1.13) consists of two pairs of 1-forms (a_j, e_j) on $\mathbb{R}^d$. We say that (a_j, e_j) is the initial data for a Yang-Mills wave A if

$$(A_j, F_{0j}) \restriction_{\{t=0\}} = (a_j, e_j).$$

Note that (1.13) for $\beta = 0$ imposes the condition that the following equation be true for any initial data for (1.13):

$$\mathbf{D}^j e_j = 0. \tag{1.14}$$

where $\mathbf{D}^j$ denotes the covariant derivative with respect to the a_j connection. This equation is the *Gauss* (or the *constraint*) *equation* for (1.13).

We observe again that harmonic Yang-Mills connections play the role of steady states for the hyperbolic Yang-Mills evolution. However, here we have an additional class of symmetries, namely the Lorentz group. Taking a Lorentz transform of a steady state yields a soliton, which evolves with constant speed less than 1. It is a simple computation to verify that the energy of this soliton is larger than the energy of the original harmonic Yang-Mills connection.

Yet again, (1.13) is gauge invariant. There are several interesting gauge choices one can make for the hyperbolic Yang-Mills equations:

— The Lorenz gauge,

$$\partial^\alpha A_\alpha = 0. \tag{1.15}$$

In this gauge the hyperbolic Yang-Mills equations become a semilinear wave system

$$\Box_A A_\alpha = -[A^\beta, \mathbf{D}_\alpha A_\beta]. \tag{1.16}$$

In particular it has finite speed of propagation. This gauge is very convenient for local well-posedness for large but regular data. Unfortunately there are multiple technical difficulties if one tries to implement such a gauge in the low regularity setting, see e.g., [**29**].

— The temporal gauge,

$$A_0 = 0. \tag{1.17}$$

This is akin to the local caloric gauge for the heat flow. In particular, at the level of the divergence of A_x we again get a pure transport equation. In this gauge the Yang-Mills system is still strictly hyperbolic, and in particular it has finite speed of propagation. Because of this, it is also convenient for local well-posedness for large but regular data. Unfortunately working with either low regularity solutions or long time solutions runs into difficulties largely caused by the lack of decay/dispersion in the transport part.

— The Coulomb gauge

$$\partial^j A_j = 0, \tag{1.18}$$

where only the spatial divergence is considered. Here the causality is lost; however, the Coulomb gauge is an "elliptic" gauge which captures well the null structure of the problem, and thus works well in low regularity settings. Indeed, the Coulomb gauge was used in [**12**] to prove the small data result for this problem in the critical Sobolev space. Unfortunately, for large data there are issues with the Coulomb gauge.

The aim of the second part of this paper will be to use the Yang-Mills heat flow in the local caloric gauge in order to introduce a new gauge choice for the hyperbolic Yang-Mills flow, which we call the *caloric gauge*. The final objective here is to arrive at a good formulation of the hyperbolic Yang-Mills equation in the caloric gauge.

1.4. Energy, scaling and criticality

Here we review the standard energy, scaling and criticality considerations which apply to the three Yang-Mills problems described above.

1.4.1. The harmonic Yang-Mills equation. — In the context of the harmonic Yang-Mills equation, the Lagrangian $\mathscr{L}_e[A]$ plays the role of the energy of the connection A; we will suggestively use the alternate notation $\mathcal{E}_{\mathbb{R}^d}[A]$ for it.

On the other hand, the Equations (1.2) also have a scale invariance property,

$$A(x) \to \lambda A(\lambda x).$$

The Sobolev space with the same scaling is $\dot{H}^{\frac{d-2}{2}}$, which we will view as the natural space for solutions to (1.2). We will refer to this space as the critical Sobolev space. The energy is scale invariant in dimension $d = 4$, which we will refer to as the *energy critical dimension.*

1.4.2. The Yang-Mills heat flow. — Here the energy plays the role of a Lyapunov functional,

$$\frac{d}{ds}\mathcal{E}_e[A] = -\int_{\mathbb{R}^d} |\mathbf{D}^j F_{jk}|^2 dx. \tag{1.19}$$

The scale invariance property now reads

$$A(x, s) \to \lambda A(\lambda x, \lambda^2 s).$$

The critical Sobolev space for the initial data a is again $\dot{H}^{\frac{d-2}{2}}$, and the energy critical dimension is $d = 4$ as well.

1.4.3. The hyperbolic Yang-Mills equation. — Here the gauge invariant energy is given by

$$\mathcal{E}_{\{t\}\times\mathbb{R}^d}[A] = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^d} \sum_{\alpha<\beta} |F_{\alpha\beta}|^2 dx.$$

The scale invariance property has the form

$$A(t, x) \to \lambda A(\lambda t, \lambda x).$$

The critical Sobolev space for the initial data (a, e) is $\dot{H}^{\frac{d-2}{2}} \times \dot{H}^{\frac{d-4}{2}}$, and the energy critical dimension is again $d = 4$.

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