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A mod p JACQUET-LANGLANDS RELATION AND SERRE FILTRATION VIA THE GEOMETRY OF HILBERT MODULAR VARIETIES: SPLICING AND DICING

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Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Fax: (33) 01 40 46 90 96
`asterisque@smf.emath.fr` • `http://smf.emath.fr/`

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Fred Diamond
Department of Mathematics, King's College London, WC2R 2LS, UK
`fred.diamond@kcl.ac.uk`

Payman Kassaei
Department of Mathematics, King's College London, WC2R 2LS, UK
`payman.kassaei@kcl.ac.uk`

Shu Sasaki
School of Mathematical Sciences, Queen Mary University of London, E1 4NS, UK
`s.sasaki.03@cantab.net`

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A mod p JACQUET-LANGLANDS RELATION AND SERRE FILTRATION VIA THE GEOMETRY OF HILBERT MODULAR VARIETIES: SPLICING AND DICING

by Fred DIAMOND, Payman KASSAEI & Shu SASAKI

Abstract. — We consider Hilbert modular varieties in characteristic p with Iwahori level at p and construct a geometric Jacquet-Langlands relation showing that the irreducible components are isomorphic to products of projective bundles over quaternionic Shimura varieties of level prime to p. We use this to establish a relation between mod p Hilbert and quaternionic modular forms that reflects the representation theory of GL_2 in characteristic p and generalizes a result of Serre for classical modular forms. Finally we study the fibers of the degeneracy map to level prime to p and prove a cohomological vanishing result that is used to associate Galois representations to mod p Hilbert modular forms.

Résumé. **(Relation de Jacquet-Langlands et filtration de Serre modulo p via la géometrie des variétés modulaires de Hilbert : épissage et découpage)** — On considère les variétés modulaires de Hilbert en caractéristique p de niveau Iwahori en p, et on construit une relation géométrique de Jacquet-Langlands qui montre que les composantes irréductibles sont isomorphes à des produits de fibrés projectifs sur des variétés de Shimura quaternioniques de niveau premier à p. On l'utilise pour établir une relation entre les formes modulaires de Hilbert et quaternioniques modulo p qui reflète la théorie des représentations de GL_2 en caractéristique p, et qui généralise un résultat de Serre pour les formes modulaires classiques. Enfin, on étudie les fibres de l'application naturelle vers la variété de niveau premier à p, et on démontre un résultat d'annulation de cohomologie qui est utilisé dans la construction des représentations galoisiennes associées aux formes modulaires de Hilbert modulo p.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Overview

Suppose that $N \geq 4$ is an integer and p is a prime not dividing N, and let $X_1(N;p)$ denote the modular curve associated to the group $\Gamma_1(N) \cap \Gamma_0(p)$. According to a fundamental result of Deligne and Rapoport [**8**, V.1], the curve $X_1(N;p)$ has a semistable model over $\mathbf{Z}[1/N]$ whose reduction mod p is a union of two irreducible components, each isomorphic to the reduction of $X_1(N)$, the modular curve associated to $\Gamma_1(N)$. The model is defined by viewing $X_1(N;p)$ as parametrizing isogenies of degree p between elliptic curves (with a point of order N), the components in characteristic p are described by whether the isogeny or its dual has connected kernel, and they cross at the points corresponding to supersingular elliptic curves. In a similar vein, they describe (in [**8**, V.2]) a semistable model over $\mathbf{Z}[\zeta_p, 1/N]$ for the modular curve $X_1(Np)$ associated to $\Gamma_1(Np)$; its structure in characteristic p underpins an elegant relation, due to Serre (see [**20**, Thm. 12.8.8] and [**16**, §8]), between mod p modular forms of weight 2 with respect to $\Gamma_1(Np)$ and those of weights ranging from 2 to $p+1$ with respect to $\Gamma_1(N)$.

The analogous situation becomes more complicated for Hilbert modular varieties, i.e., the Shimura varieties associated to $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$ where F is a totally real number field. Recall that these varieties have dimension $d = [F : \mathbf{Q}]$, and can be viewed as (coarse) moduli spaces for certain d-dimensional abelian varieties with additional structure. We restrict our attention to the case where p is unramified in F, and consider the setting, analogous to the one above, of Hilbert modular varieties with Iwahori level at p. In [**30**], Pappas defined models for these varieties over $\mathbf{Z}_{(p)}$, which he proved were flat local complete intersections of relative dimension d. The geometry of their reduction mod p has been studied by various authors, as will be discussed below. In brief, some collections of components can be identified with reductions of Hilbert modular varieties of level prime to p, as in the classical case, but there are also "intermediate" components with no such description. Our first main result is a geometric Jacquet-Langlands relation that describes them as products of projective bundles over quaternionic Shimura varieties of level prime to p.

In **[18]** Helm proves results of a similar nature in the case of unitary Shimura varieties which imply the following: collections of components in a Hilbert modular variety with Iwahori level at p are related via *Frobenius factors* to a product of projective bundles over quaternionic Shimura varieties of level prime to p. While some of our geometric methods are inspired by Helm's work, we would like to point out an essential difference in the results: the quaternion algebras appearing in our work are *different*, leading to the existence of isomorphisms on the nose (as opposed to Frobenius factors). Apart from facilitating new applications, we consider the merit of these results to be the naturality of the relationships they establish between the mod p geometry of Shimura varieties associated to different reductive groups.

There is also an essential difference between our method and the one in **[18]**: our construction of the above-mentioned isomorphisms is more direct in that it does not involve the degeneracy (or forgetful) map to the prime-to-p level. Indeed, the image of a point corresponding to an isogeny of abelian varieties is constructed by "splicing" the Dieudonné modules of the isogeny's source and target. As a result, we believe our method to be more amenable to generalization to Shimura varieties associated to higher rank groups.

Our second main set of results concerns a generalization of Serre's relation between mod p modular forms of weight 2 and level Np and those of weight $k \in [2, p+1]$ and level prime to p. More precisely, we obtain a filtration on the space of mod p Hilbert modular forms of parallel weight 2 and pro-p Iwahori level at p, and identify the graded pieces with spaces of quaternionic modular forms of level prime to p and weight (components) in $[2, p+1]$. We accomplish this by combining our geometric Jacquet-Langlands relation with an analysis of dualizing sheaves by a method we call "dicing". The motivation for the result, discussed further below, comes from the relation between algebraic and geometric Serre weights explored in **[10]**. In fact, this relation was a critical clue to our understanding of the more canonical choice of quaternion algebras.

Our final set of geometric results centers on the degeneracy map from the Hilbert modular variety of Iwahori level at p to level prime to p (i.e., the one intervening in Helm's approach as opposed to ours). In particular, we apply techniques from crystalline Dieudonné theory to determine the precise structure of its fibers (restricted to irreducible components). These results complement those given in the Key Lemma in **[15]**, and are expected to have applications in the context of that work. We give a different application in this paper: combining this with our method of dicing dualizing sheaves, we prove a cohomological vanishing result which is a key ingredient in the construction in **[10]** of Galois representations associated to (non-paritious) Hilbert modular eigenforms in characteristic p.

1.2. Geometric Jacquet-Langlands

We first introduce the notation for our main objects of study: certain mod p Shimura varieties and automorphic bundles. Their precise definitions are given in §§2–4, along with various technical results that will be needed later in the paper.

We fix a totally real field F and a prime p unramified in F. We also fix embeddings $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ and $\overline{\mathbf{Q}} \to \mathbf{C}$, and let Θ denote the set of embeddings $F \to \overline{\mathbf{Q}}$. We can thus identify Θ with the sets of homomorphisms:

$$\Theta_\infty := \{F \to \mathbf{R}\}, \quad \Theta_p := \{F \to \overline{\mathbf{Q}}_p\} \quad \text{and} \quad \overline{\Theta}_p := \{O_F/p \to \overline{\mathbf{F}}_p\}.$$

We let ϕ denote the Frobenius automorphism of $\overline{\mathbf{F}}_p$. Furthermore, we have a natural decomposition $\overline{\Theta}_p = \coprod_{v|p} \Theta_v$, where $\theta \in \Theta_v$ if and only if θ factors through $\mathcal{O}_F/v$, and each Θ_v is an orbit in $\overline{\Theta}_p$ under $\theta \mapsto \phi \circ \theta$.

Let $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$, and let U be a sufficiently small open compact subgroup of $G(\mathbf{A}_\mathbf{f}) = \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. The Hilbert modular variety with complex points

$$\mathrm{GL}_2(F)\backslash((\mathbf{C} - \mathbf{R})^\Theta \times \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}))/U$$

has a canonical integral model over $\mathbf{Z}_{(p)}$, which we denote $Y_U(G)$, and write $\overline{Y} = Y_U(G) \times_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$ for its geometric special fiber. Similarly for any non-empty $\Sigma \subset \Theta$ of even cardinality, consider the quaternion algebra $B = B_\Sigma$ over F ramified at precisely the set of infinite places corresponding to Σ, and let G_Σ denote the algebraic group over $\mathbf{Q}$ defined by $G_\Sigma(R) = (B \otimes_\mathbf{Q} R)^\times$. Choosing a maximal order $\mathcal{O}_B$ of B and an isomorphism $\widehat{\mathcal{O}}_B \cong M_2(\widehat{\mathcal{O}}_F)$, we can identify U with an open compact subgroup of $G_\Sigma(\mathbf{A}_\mathbf{f}) \cong G(\mathbf{A}_\mathbf{f})$ and consider the quaternionic Shimura variety with complex points

$$B^\times\backslash((\mathbf{C} - \mathbf{R})^{\Theta - \Sigma} \times (B \otimes \mathbf{A}_\mathbf{f})^\times)/U.$$

We let $Y_U(G_\Sigma)$ denote its canonical integral model, defined over the localization at a prime over p in its reflex field; denote its geometric special fiber by $\overline{Y}_\Sigma$, and let $\overline{Y}_\emptyset = \overline{Y}$. Thus $\overline{Y}_\Sigma$ is a smooth variety over $\overline{\mathbf{F}}_p$ of dimension $|\Theta - \Sigma|$, which is proper if and only if $\Sigma \neq \emptyset$.

For each $\theta \in \Theta$, one can also define a rank two automorphic vector bundle $\mathcal{V}_\theta$ on $\overline{Y}_\Sigma$, together with a line bundle $\omega_\theta \subset \mathcal{V}_\theta$ whenever $\theta \notin \Sigma$. We let δ_θ denote the line bundle $\bigwedge^2_{\mathcal{O}_{\overline{Y}_\Sigma}} \mathcal{V}_\theta$ on $\overline{Y}_\Sigma$. For $(k,\ell) \in \mathbf{Z}^\Theta \times \mathbf{Z}^\Theta$ with $k_\theta \geq 2$ for all $\theta \in \Sigma$, we define the automorphic bundle of weight (k,ℓ) on $\overline{Y}_\Sigma$ as

$$\mathscr{A}_{k,\ell} = \left(\bigotimes_{\theta \notin \Sigma} \delta_\theta^{\ell_\theta} \omega_\theta^{k_\theta} \right) \bigotimes \left(\bigotimes_{\theta \in \Sigma} \delta_\theta^{\ell_\theta} \mathrm{Sym}^{k_\theta - 2} \mathcal{V}_\theta \right).$$

The space of mod p modular forms of weight (k,ℓ) and level U with respect to G_Σ is then defined as $H^0(\overline{Y}_\Sigma, \mathscr{A}_{k,\ell})$, under the assumption $F \neq \mathbf{Q}$. (For $F = \mathbf{Q}$, one has to extend the line bundle to the cusps in order to recover the usual notion.) The bundles are equipped with a natural action of $G_\Sigma(\mathbf{A}_{\mathbf{f}}^{(p)})$ defined compatibly with its action on the varieties $\overline{Y}_\Sigma$ for varying U, yielding a Hecke action on the spaces of forms.

Restrict attention now to the case $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ and consider the open compact subgroup

$$U_0(p) = \{g \in U \mid g_p \equiv (\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}) \bmod p\mathcal{O}_{F,p}\}.$$

We let $Y_{U_0(p)}(G)$ denote the model over $\mathbf{Z}_{(p)}$ defined by Pappas in [**30**] for the Hilbert modular variety of level $U_0(p)$, and let $\overline{Y}_0(p)$ denote its geometric special fiber. Pappas studied the local structure of $\overline{Y}_0(p)$ and proved that it is a flat local complete intersection of relative dimension $d = [F : \mathbf{Q}]$. The global geometry of $\overline{Y}_0(p)$ and its degeneracy map to $\overline{Y}$ were studied in [**36**] in the case $d = 2$, and for general d in [**15**], through the introduction of a stratification. A similar stratification was considered in [**18**] in the context of related unitary Shimura varieties, and a generalization to the context of Hilbert modular varieties with p ramified in F was studied in [**12**].

The stratification on $\overline{Y}_0(p)$ is given by closed subvarieties indexed by pairs (I, J) of subsets of $\Theta = \overline{\Theta}_p$ satisfying $(\phi^{-1}I) \cup J = \Theta$. The d-dimensional strata have the form $\overline{Y}_0(p)_{I,J}$ where $I = \{\phi \circ \theta \mid \theta \notin J\}$, which we denote simply by $\overline{Y}_0(p)_J$. Thus $\overline{Y}_0(p) = \bigcup_{J \subset \Theta} \overline{Y}_0(p)_J$ where each $\overline{Y}_0(p)_J$ is a smooth d-dimensional variety over $\overline{\mathbf{F}}_p$, and each irreducible component of $\overline{Y}_0(p)$ lies in $\overline{Y}_0(p)_J$ for a unique $J \subset \Theta$. We can then state our geometric Jacquet-Langlands relation (proved in §5) as follows:

Theorem A. — *For each $J \subset \overline{\Theta}_p$ and sufficiently small open compact subgroup U of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(O_{F,p})$, there is a Hecke-equivariant isomorphism*

$$\overline{Y}_0(p)_J \xrightarrow{\sim} \prod_{\theta \in \Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta),$$

where the product is a fiber product over $\overline{Y}_\Sigma$, and $\Sigma = \Sigma_J \subset \Theta_\infty$ corresponds under the identification $\Theta_\infty = \overline{\Theta}_p$ to $\{\theta \in J \mid \phi \circ \theta \notin J\} \cup \{\theta \notin J \mid \phi \circ \theta \in J\}$.

As we mentioned earlier, a similar result was proved in the context of related unitary Shimura varieties by Helm ([**18**, Thm. 5.10]) though the morphism constructed by Helm is only proved to be bijective on points, i.e., a "Frobenius factor" in the terminology of [**18**]. Using $'$ to denote the analogous unitary Shimura varieties, Helm's approach is to relate the $\overline{Y}'_0(p)_J$ to products of $\mathbf{P}^1$-bundles over lower-dimensional strata of $\overline{Y}'$, and to relate those in turn to products of $\mathbf{P}^1$-bundles over lower-dimensional Shimura varieties. We had initially hoped to use Helm's result for the application we had in mind, despite the presence of Frobenius factors in his construction. Indeed we were encouraged by the fact that results of Tian-Xiao [**38**] remove the Frobenius factor from the latter step. While the Frobenius factor is intrinsic to the former step (see Theorem D below), a more serious problem was that the set of ramified places for the quaternion algebra provided by the results in [**18**] does not match the set Σ_J determining the vector bundles $\mathcal{V}_\theta$. This led us to the consideration of different quaternion algebras: in fact the ones that emerge naturally from our method of "splicing" described below, which is more direct and bypasses the projections to strata of $\overline{Y}'$.

To prove Theorem A, we first prove the analogous result in the context of related unitary Shimura varieties (as in [**38**]), so that the quaternionic Shimura varieties are replaced by ones which are moduli spaces for abelian varieties. Denoting the special fibers of the corresponding unitary Shimura varieties by $\overline{Y}'_0(p)$ and $\overline{Y}'_\Sigma$, the stratum $\overline{Y}'_0(p)_J$ parametrizes p-isogenies $A \to B$ such that the induced morphism on Dieudonné modules satisfies conditions determined by J. The idea is to define morphisms to (projective bundles over) $\overline{Y}'_\Sigma$ by splicing the Dieudonné modules of A and B to obtain an abelian variety C corresponding to a point of $\overline{Y}'_\Sigma$. We then prove this yields an isomorphism analogous to the one we want, and explain how to transfer the result to the Hilbert/quaternionic setting to obtain Theorem A using results in §2. We remark that a key idea that enables us to obtain such clean results in comparison to [**18**] and [**38**] lies in exploiting the possibility of allowing A and B to play symmetric roles.

1.3. The Serre filtration

We now describe in more detail the application of Theorem A we had in mind, and carried out in §6. Recall that we wish to generalize a result of Serre relating mod p modular forms of weight 2 and level $\Gamma_1(Np)$ to mod p modular forms of weights $k \in [2, p+1]$ and level $\Gamma_1(N)$ (see [**20**, Thm. 12.8.8]). More precisely, let $X_1(Np)$ denote the semistable model over $R = \mathbf{Z}_{(p)}[\mu_p]$ for the compact modular curve $X_1(Np)$ (as in [**16**, §7]) and let $\mathscr{K}$ denote its dualizing sheaf. Then $H^0(X_1(Np), \mathscr{K})$ is a lattice over R in the space of weight two cusp forms with respect to $\Gamma_1(Np)$, and tensoring over R with $\overline{\mathbf{F}}_p$ yields $H^0(\overline{X}_1(Np), \overline{\mathscr{K}})$, where $\overline{X}_1(Np)$ is the special fiber of $X_1(Np)$ and its dualizing sheaf $\overline{\mathscr{K}}$ is its sheaf of regular (Rosenlicht) differentials. The space $H^0(\overline{X}_1(Np), \overline{\mathscr{K}})$ decomposes as a direct sum of eigenspaces with respect to the natural action of $(\mathbf{Z}/p\mathbf{Z})^\times$. Writing the characters $(\mathbf{Z}/p\mathbf{Z})^\times \to \overline{\mathbf{F}}_p^\times$ as $\chi_m : a \mapsto a^m$ for $m = 1, 2, \ldots, p-1$, Serre's result, as refined by Gross (see Propositions 8.13 and 8.18 of [**16**]), gives a Hecke-equivariant exact sequence

$$0 \to H^0(\overline{X}, \delta^m \omega^{p+1-m}(-C)) \to H^0(\overline{X}_1(Np), \overline{\mathscr{K}})^{\chi_m} \to H^0(\overline{X}, \omega^{m+2}(-C)) \to 0,$$

where $\overline{X}$ is the reduction of $X_1(N)$ and $\omega^k(-C)$ is the line bundle whose sections are (mod p) cusp forms of weight k with respect to $\Gamma_1(N)$, and δ is a trivial bundle whose presence has the effect of twisting the action of the Hecke operator T_q by q.

The above exact sequence can be viewed as a geometric counterpart to the one arising in the cohomology of the modular curve $\Gamma_1(N)\backslash\mathfrak{H}$ with coefficients in local systems associated to the right $\mathrm{GL}_2(\mathbf{F}_p)$-modules in the exact sequence:

$$0 \to \det{}^m \otimes \mathrm{Sym}^{p-1-m}\mathbf{F}_p^2 \to \mathrm{Ind}_P^{\mathrm{GL}_2(\mathbf{F}_p)}(1 \otimes \chi_m) \to \mathrm{Sym}^m \mathbf{F}_p^2 \to 0,$$

where P is the subgroup of upper-triangular matrices and $1 \otimes \chi_m$ is the character sending $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\chi_m(a) = d^m$. From the point of view of Serre weight conjectures for Galois representations, and in particular the relation between algebraic and geometric notions of Serre weights as explored in [**10**], it is natural to seek a generalization of Serre's result to the context of Hilbert modular forms. The desired result should describe the space of mod p forms of parallel weight 2 and character χ with respect to pro-p-Iwahori level at p in terms of spaces of mod p forms of level prime to p and weights corresponding to the Jordan-Holder factors of the right representation $\mathrm{Ind}_P^{\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_p)}\chi$, where P is again the subgroup of upper-triangular matrices and $\chi : P \to \overline{\mathbf{F}}_p^\times$ is any character, which by twisting easily reduces to the case of characters the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto \chi(d)$ where χ is a character on $(\mathcal{O}_F/p\mathcal{O}_F)^\times$.

To that end, we maintain the notation from the discussion before Theorem A and now consider

$$U_1(p) = \{g \in U \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p\mathcal{O}_{F,p}\}.$$

Then Pappas provides us with a model for $Y_{U_1(p)}(G)$ which is finite and flat over $Y_{U_0(p)}(G)$, and hence Cohen-Macaulay over $\mathbf{Z}_{(p)}$. Let $\overline{Y}_1(p)$ denote its geometric special fiber, and let $\mathscr{K}$ (resp. $\overline{\mathscr{K}}$) denote the dualizing sheaf on $Y_{U_1(p)}(G)$ (resp. $\overline{Y}_1(p)$). Via the Kodaira-Spencer isomorphism $H^0(Y_{U_1(p)}(G), \mathscr{K})$ can be identified with a lattice over $\mathbf{Z}_{(p)}$ in the space of Hilbert modular forms of parallel weight 2 and level $U_1(p)$ over $\mathbf{Q}$, and we view $H^0(\overline{Y}_1(p), \overline{\mathscr{K}})$ as the space of mod p forms of parallel weight 2 and level $U_1(p)$. The natural action of the group $U_0(p)/U_1(p) \cong (\mathcal{O}_F/p\mathcal{O}_F)^\times$ on $H^0(\overline{Y}_1(p), \overline{\mathscr{K}})$ yields a decomposition

$$H^0(\overline{Y}_1(p), \overline{\mathscr{K}}) = \bigoplus_\chi H^0(\overline{Y}_1(p), \overline{\mathscr{K}})^\chi$$

into eigenspaces for the characters $\chi : (\mathcal{O}_F/p\mathcal{O}_F)^\times \to \overline{\mathbf{F}}_p^\times$.

Before stating the second main result, we recall two basic facts from the representation theory of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$. Firstly, its irreducible (right) representations over $\overline{\mathbf{F}}_p$ are precisely those of the form:

$$V_{m,n} = \bigotimes_{\theta \in \overline{\Theta}_p} \det{}^{m_\theta} \otimes \mathrm{Sym}^{n_\theta} V_\theta$$

for $(m, n) \in \mathbf{Z}^\Theta \times \mathbf{Z}^\Theta$ with $0 \le m_\theta, n_\theta \le p-1$ for all θ and $m_\theta < p-1$ for some θ in each Θ_v, and $V_\theta = \overline{\mathbf{F}}_p^2$ with $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ acting via θ. Secondly (see [**6**, §2]), there is a decreasing filtration

$$0 \subset \mathrm{Fil}^d V_\chi \subset \mathrm{Fil}^{d-1} V_\chi \subset \cdots \subset \mathrm{Fil}^1 V_\chi \subset \mathrm{Fil}^0 V_\chi = V_\chi$$

on the representation $V_\chi = \mathrm{Ind}_P^{\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_p)}\chi$ such that the graded part $\mathrm{gr}^j V_\chi$ has the form $\bigoplus_{|J|=j} V_{\chi,J}$, where each $V_{\chi,J}$ is either irreducible or zero. We then prove the following theorem in §6.3.3 (see §6.3.4 for elaboration on the meaning of Hecke-equivariance):

Theorem B. — *For each sufficiently small open compact subgroup U of* $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ *containing* $\mathrm{GL}_2(O_{F,p})$*, there is a Hecke-equivariant spectral sequence*

$$E_1^{j,i} = \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_{\Sigma_J}, \mathscr{A}_{\chi,J}) \Longrightarrow H^{i+j}(\overline{Y}_1(p), \overline{\mathscr{K}})^{\chi},$$

where $\mathscr{A}_{\chi,J} = \mathscr{A}_{n+2,m}$ *(resp.* 0*) if* $V_{\chi,J} \cong V_{m,n}$ *(resp.* 0*).*

We thus obtain the following generalization of Serre's filtration, where the graded pieces take the same form as in the classical case, except that spaces of Hilbert modular forms are in general replaced by the quaternionic ones to which they correspond via Jacquet-Langlands.

Corollary C. — *For each sufficiently small open compact subgroup U of* $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ *containing* $\mathrm{GL}_2(O_{F,p})$*, there is a Hecke-equivariant decreasing filtration of length $d+1$ on* $H^0(\overline{Y}_1(p), \overline{\mathscr{K}})^{\chi}$*, together with a Hecke-equivariant inclusion:*

$$\mathrm{gr}^j\left(H^0(\overline{Y}_1(p), \overline{\mathscr{K}})^{\chi}\right) \hookrightarrow \bigoplus_{|J|=j} H^0(\overline{Y}_{\Sigma_J}, \mathscr{A}_{\chi,J})$$

for $j = 0, 1, \ldots, d$.

In order to prove Theorem B, we consider the direct image of $\mathscr{K}$ under the projection $\overline{Y}_1(p) \to \overline{Y}_0(p)$, decompose it into line bundles $\mathscr{K}_\chi$ on $\overline{Y}_0(p)$ under the action of $U_0(p)/U_1(p)$, and define a filtration on $\mathscr{K}_\chi$ by restricting to the strata. Using the fact that the $\overline{Y}_0(p)_J$ are obtained by successively bisecting $\overline{Y}_0(p)$ into local complete intersections, we obtain a description of the graded pieces of the filtration in terms of line bundles $\mathscr{K}_{\chi,J}$ on the $\overline{Y}_0(p)_J$. Our method of "dicing" the dualizing sheaf may be of independent interest, and is used again in the proof of Theorem E below. The proof of Theorem B is then completed by determining the line bundles to which the $\mathscr{K}_{\chi,J}$ correspond under the isomorphism of Theorem A.

1.4. Degeneracy fibers

Our final set of results, the subject of §7, concerns the degeneracy map $\overline{Y}_0(p) \to \overline{Y}$, or more precisely, its restriction to the stratum $\overline{Y}_0(p)_J$ (maintaining the above notation). This restriction is known (see **[15]**) to factor through a pointwise bijective morphism ξ_J from $\overline{Y}_0(p)_J$ to a product of $\mathbf{P}^1$-bundles over a lower-dimensional stratum in $\overline{Y}$; thus ξ_J is a Frobenius factor in the sense of **[18]**. We make this more precise by showing that ξ_J is a factor of the Frobenius itself (rather than a power), and we go on to determine the precise structure of the fibers of $\overline{Y}_0(p)_J \to \overline{Y}$. In particular we prove the following (see Theorem 7.2.3.2 and the subsequent discussion for an even more precise version):

Theorem D. — *If Z is a non-empty fiber of the morphism $\overline{Y}_0(p)_J \to \overline{Y}$ over a closed point of $\overline{Y}$, then Z is isomorphic to* $(\mathbf{P}^1_{\overline{\mathbf{F}}_p})^r \times (\mathrm{Spec}(\overline{\mathbf{F}}_p[t]/(t^p)))^s$*, where* $r = |\Sigma_J|/2$ *and* $s = |J| - r$.

Our approach to proving Theorem D relies heavily on crystalline Dieudonné theory. In particular, we use the full faithfulness of the Dieudonné crystal functor over smooth bases, due to Berthelot-Messing [**3**], in order to obtain the Frobenius factorization, which we then use to determine the local structure of the fiber Z. In order to determine the global structure of Z, we show that certain pointwise relations between the Dieudonné modules of A and B (where $A \to B$ is a universal isogeny) in fact arise from isomorphisms of crystals over Z.

We remark that the problem of describing the fibers of $\overline{Y}_0(p)_J \to \overline{Y}$ is also considered in [**12**, §4.9], where a weaker result than Theorem D is used in their approach to constructing Hecke operators at primes dividing p. Related results, complementary to ones in this paper, are obtained in [**15**] where the degeneracy morphism is studied before restriction to the strata. Further motivation for such analysis of the degeneracy map is provided by its applications to p-adic analytic continuation of Hilbert modular forms via the dynamics of Hecke operators at primes over p, as in [**19**] and [**34**].

In this paper, we combine (the more precise version of) Theorem D with the method of dicing introduced in §6 [1] in order to prove the following cohomological vanishing result.

Theorem E. — *If $\pi : Y_{U_1(p)}(G) \to Y_U(G)$ is the natural projection (of models over $\mathbf{Z}_{(p)}$) and $\mathscr{K}$ is the dualizing sheaf on $Y_{U_1(p)}(G)$, then $R^i\pi_*\mathscr{K} = 0$ for all $i > 0$.*

We note that Theorem E (for $i = 1$) is a crucial ingredient in the proof of Theorem 6.1.1 of [**10**], which associates Galois representations to mod p Hilbert modular eigenforms of arbitrary weight.

1.5. Questions

We close the Introduction by listing several questions and directions for further research that are suggested by our work.

Question 1. — Is there a more general framework for Theorems A and B where the group $G = \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_2$ is replaced by the one associated to a quaternion algebra over F (or an even more general reductive group), and the representation $\mathrm{Ind}\,\chi$ is replaced by any tamely ramified (or even more general) type? For a totally definite quaternion algebra, where the associated Shimura varieties are zero-dimensional, the analogues of the theorems are essentially tautologies, while the case of Shimura curves is related to the work of Newton-Yoshida [**26**].

Question 2. — The flipping and twisting of weights that appear in the computation of the line bundles in the proof of Theorem B perfectly reflect the same phenomena in the computation of the Jordan-Hölder factors of the principal series types. Can one give a more conceptual explanation for this synchronized gymnastics?

1. The results of §7 are, however, independent of those in §5.

Question 3. — Note that Corollary C only produces an injection. The obstruction to proving that it is an isomorphism comes from terms of the form $H^1(\overline{Y}_{\Sigma_J}, \mathscr{A}_{\chi,J})$ in the spectral sequence, and one can construct examples where these do not vanish. If there is in fact a non-trivial cokernel, can one at least prove that the Hecke action on it is "Eisenstein"?

Question 4. — A Hilbert modular eigenform f of parallel weight 2 and level $U_1(p)$, with coefficients in a finite extension $\mathscr{O}$ of $\mathbf{Z}_p$, determines a rank one submodule of the space of sections of the dualizing sheaf on $Y_{U_1(p)}(G)$ over $\mathscr{O}$, and hence a one-dimensional subspace of $H^0(\overline{Y}_1(p), \mathscr{K})^\chi$ for some χ. Motivated by the conjectures of [**10**], one can ask if its position in the filtration and inclusion of Corollary C is determined by the local Galois representations [2] $\rho_f|_{G_{F_v}}$ for $v|p$, or more precisely the invariants v_θ for $\theta \in \Theta_v$ associated to $\rho_f|_{G_{F_v}}$ as in the formulation of Breuil's Lattice Conjecture [**5**, Conj. 1.2] (proved by Emerton-Gee-Savitt [**11**]).

1.6. Acknowledgments

The authors would like to thank D. Helm for several useful conversations. We have already described how our results on geometric Jacquet-Langlands are inspired by those in his paper [**18**], but we should also remark that this circle of ideas has deeper roots in Zink's work [**40**], Serre's letters [3] [**35**], Ribet's seminal paper [**33**], Pappas's thesis [**29**] and work of Ghitza [**14**] and Nicole [**27**]. It is a pleasure to acknowledge that some of the seeds for this paper were in fact planted by Pappas's description of the results in his thesis to one of the authors (F.D.) at Columbia in the 1990's; they were only germinated in the last few years by ideas diffusing from the geometric Serre weight conjectures in [**10**].

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2. Together with the U_v-eigenvalue if f is old at v.
3. Written in 1987 and 1989.