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BROWNIAN STRUCTURE IN THE KPZ FIXED POINT

by Jacob CALVERT, Alan HAMMOND & Milind HEGDE

Abstract. — Many models of one-dimensional local random growth are expected to lie in the Kardar-Parisi-Zhang (KPZ) universality class. For such a model, the interface profile at advanced time may be viewed in scaled coordinates specified via characteristic KPZ scaling exponents of one-third and two-thirds. When the long time limit of this scaled interface is taken, it is expected—and proved for a few integrable models—that, up to a parabolic shift, the Airy_2 process $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ is obtained. This process may be embedded via the Robinson-Schensted-Knuth correspondence as the uppermost curve in an $\mathbb{N}$-indexed system of random continuous curves, the Airy line ensemble.

Among our principal results is the assertion that the Airy_2 process enjoys a very strong similarity to Brownian motion B (of rate two) on unit-order intervals. This result yields bounds on the Airy_2 probabilities of a large class of events from the counterpart bounds on Brownian motion probabilities. The result has the consequence that the Radon-Nikodym derivative of the law of $\mathscr{A}$ on say $[-1, 1]$ after a suitable vertical shift, with respect to the law of B on the same interval, lies in every L^p space for $p \in (1, \infty)$. In fact, the quantitative comparison of probability bounds we prove also holds for the scaled energy profile with Dirac delta initial condition of the model of Brownian last passage percolation, a model that lies in the KPZ universality class and in which the energy of paths in a random Brownian environment is maximized.

Our technique of proof harnesses a probabilistic resampling or *Brownian Gibbs* property satisfied by the Airy line ensemble after parabolic shift, and this article develops Brownian Gibbs analysis of this ensemble begun in work of Corwin and Hammond (2014) and pursued by Hammond (2019). Our Brownian comparison for scaled interface profiles is an element in the ongoing program of studying KPZ universality via probabilistic and geometric methods of proof, aided by limited but essential use of integrable inputs. Indeed, the comparison result is a useful tool for studying this universality class. We present and prove several applications, concerning for example the structure of near ground states in Brownian last passage percolation, or Brownian structure in scaled interface profiles that arise from the evolution from any element in a very general class of initial data.

Résumé. **(Structure brownienne dans le point fixe KPZ)** — On s'attend à ce que de nombreux modèles de croissance aléatoire uni-dimensionelle appartiennent à la classe d'universalité de Kardar-Parisi-Zhang (KPZ). En changeant convenablement le temps d'échelle via les exposants caracteristiques de KPZ d'un tiers et de deux tiers, on considère le profil determiné par un tel modèle. Dans la limite de temps infini, ce profil devrait converger vers le processus Airy_2 $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ à l'addition d'une parabole près; constatation dont la preuve est connue dans les cas de certains modèles integrables. Ce processus peut-être identifié par la correspondance de Robinson-Schensted-Knuth à la courbe la plus haute d'un système indexé par $\mathbb{N}$ de courbes aléatoires continues: l'ensemble de lignes d'Airy. Parmi nos résultats principaux est l'énoncé que le processus Airy_2 présente une grande similarité avec le mouvement brownien (de taux deux) sur un intervalle de longueur de l'ordre de l'unité. Grâce à ce résultat, nous obtenons des bornes sur les probabilités Airy_2 d'une large classe d'évenements à partir de bornes browniennes homologues. Ce résultat a le corollaire que la derivée Radon-Nikodym de la loi $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ sur l'intervalle $[-1, 1]$ par rapport à la loi brownienne sur cet intervalle appartient à tous les espaces L^p où $p \in (1, \infty)$. En fait, la comparaison quantitative de bornes probabilistiques est également vraie pour le profil d'énergie après changement d'échelle (avec condition initiale delta Dirac) du modèle de percolation de passage dernier brownien, un modele qui appartient à la classe d'universalité de KPZ et dans lequel l'énergie de chemins est maximisée dans un environnement aléatoire brownien. Notre technique de preuve utilise une propriété de rééchantillonnage probabiliste, la propriété de Gibbs brownienne, qui est satisfaite par l'ensemble parabolique de lignes d'Airy; cet article développe l'analyse de Gibbs brownienne de cet ensemble commencée par Corwin et Hammond (2014) et poursuivie par Hammond (2019). Notre comparaison brownienne de profils après changement d'échelle est un élément du programme plus vaste de l'étude de l'universalité de KPZ par les moyens géométriques et probabilistiques, facilité par l'usage, limité mais critique, des énoncés integrables. En effet, le résultat de comparaison est un outil puissant dans l'étude de cette classe d'universalité. Nous prouvons quelques corollaires au sujet de la structure d'états quasi-minimaux dans le percolation de dernier passage brownien, ou de la structure brownienne trouvée dans les profils d'interface après changement d'échelle donnée par l'évolution d'un élément arbitraire dans une classe très large de conditions initiales.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. KPZ universality

The field of Kardar-Parisi-Zhang (KPZ) universality concerns one-dimensional interfaces that evolve randomly in time, and universal random structures that describe, independently of the microscopic details that specify the local evolution of such random models, the geometry and fluctuation of the interface when time is advanced. The KPZ universality class, whose members are random processes that are expected to evince these late-time characteristics, is very broad. The basic features of a random growth model that indicate that it may be a member of the KPZ universality class are that growth occurs in a direction normal to the present local slope of the interface at a rate influenced by the slope, alongside two competing forces: a smoothing effect generated by surface tension, and a roughening effect caused by forces in the environment that are local and random.

The principal results of this monograph offer a very strong assertion concerning the geometry of the canonical random object—*the* Airy_2 *process with parabolic curvature*—that describes the scaled attributes of what are known as the narrow wedge solutions to models in the KPZ universality class. Our results further offer a counterpart description that holds in a rather uniform sense in the prelimit for a model in the KPZ class known as *Brownian last passage percolation* (LPP). The geometric inference that we make is a powerful assertion of the Brownian nature of the scaled interface profile and it is the engine for a wide array of applications about scaled random growth models and their KPZ universality limiting structure.

The parabolically shifted Airy_2 process is the first in a family of limiting processes, indexed by the initial condition, which are expected to be universal objects in the KPZ universality class, and which may be referred to collectively as the *KPZ fixed point*. The name arises from the physical view that such objects are expected to be fixed points of suitable renormalization operators. (Indeed, an important related object has been constructed by **[38]** in a paper bearing the name "The KPZ fixed point". Our usage of this term is a corruption of theirs, but only mildly so. We will comment further on **[38]** in Section 1.3.2.) One application of our results will be a statement about a certain form of Brownian regularity for these KPZ fixed point profiles, i.e., limiting scaled interface profiles from general initial data in Brownian LPP.

We defer the definition of Brownian last passage percolation and the presentation of our main theorem, expressed in prelimiting terms that capture Brownian LPP, to Section 3.2. What we are able to indicate in the ensuing paragraphs is the form of our principal assertion in the limiting case of scaled KPZ structure. Indeed, we will next introduce the parabolic Airy_2 process; and then we state our principal conclusion as it applies to this process.

The Airy_2 process $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ is a stationary process first introduced by Prähofer and Spohn **[47]** in a scaled description of the polynuclear growth (PNG) model; or equivalently, of another famous last passage percolation model, Poissonian LPP. It was defined in **[47]** in terms of its finite-dimensional distributions—written via determinantal formulas involving the Airy kernel—and shown to have a continuous version. Its basic role in KPZ universality may be expressed via the scaled "narrow wedge" interface $L_n : \mathbb{R} \to \mathbb{R}$ for various KPZ models, where n is an abstract parameter measuring the size of the system in the model. It is a major open problem to rigorously show in a broad class of models that the distributional high-n limit, with respect to the topology of locally uniform convergence on the space of continuous functions with domain and co-domain the real line, of the process $x \mapsto L_n(x)$ (after adopting model-dependent constants in the scalings) is $\mathscr{A}(x) - x^2$, a parabolically shifted Airy_2 process.

This inference has recently been proven for the KPZ equation **[50, 55]**, and had been validated for several stochastic growth models with narrow wedge initial data beyond PNG. These models include the totally asymmetric simple exclusion process (TASEP) **[8]** as well as the model that will be the principal object of rigorous attention in this monograph, namely Brownian LPP. In the latter case, the convergence is proved via a distributional relation with Dyson Brownian motion that will be reviewed in Section 3.2.

1.1.1. Locally Brownian nature of the limiting process. — In the case of Brownian LPP, as mentioned, the limiting process is $\mathscr{A}(x) - x^2$. We define

$$\mathscr{L}(x) := 2^{-1/2} \left(\mathscr{A}(x) - x^2 \right),$$

and call it the *parabolic* Airy_2 *process*, in spite of the factor of $2^{-1/2}$. The factor $2^{-1/2}$ is included to make comparisons with Brownian motion more convenient and will be made clearer momentarily.

The limiting process globally adopts a parabolic form, but it is locally Brownian—see Figure 1.1. The term "locally Brownian" may be interpreted in several ways, with a progression to stronger forms of interpretation, reflecting recent progress in understanding this limiting scaled profile. "Locally Brownian" could mean that, for any given $x \in \mathbb{R}$, the distributional process limit of $y \mapsto \varepsilon^{-1/2}(\mathscr{L}(x + \varepsilon y) - \mathscr{L}(x))$ as $\varepsilon \searrow 0$ is a standard Brownian motion (where it is the presence of the factor $2^{-1/2}$ in the definition of $\mathscr{L}$ that permits the diffusion rate to equal one, as $\mathscr{A}$ itself is locally