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ELLIPTIC THEORY IN DOMAINS WITH BOUNDARIES OF MIXED DIMENSION

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ELLIPTIC THEORY
IN DOMAINS WITH BOUNDARIES OF MIXED DIMENSION

by Guy DAVID, Joseph FENEUIL & Svitlana MAYBORODA

Abstract. — Take an open domain $\Omega \subset \mathbb{R}^n$ whose boundary may be composed of pieces of different dimensions. For instance, Ω can be a ball on $\mathbb{R}^3$, minus one of its diameters D, or a so-called saw-tooth domain, with a boundary consisting of pieces of 1-dimensional curves intercepted by 2-dimensional spheres. It could also be a domain with a fractal (or partially fractal) boundary. Under appropriate geometric assumptions, essentially the existence of doubling measures on Ω and $\partial\Omega$ with appropriate size conditions—we construct a class of second order degenerate elliptic operators L adapted to the geometry, and establish key estimates of elliptic theory associated to those operators. This includes boundary Poincaré and Harnack inequalities, maximum principle, and Hölder continuity of solutions at the boundary. We introduce Hilbert spaces naturally associated to the geometry, construct appropriate trace and extension operators, and use them to define weak solutions to $Lu = 0$. Then we prove De Giorgi-Nash-Moser estimates inside Ω and on the boundary, solve the Dirichlet problem and thus construct an elliptic measure ω_L associated to L. We construct Green functions and use them to prove a comparison principle and the doubling property for ω_L. Since our theory emphasizes measures, rather than the geometry per se, the results are new even in the classical setting of a half-plane $\mathbb{R}^2_+$ when the boundary $\partial\mathbb{R}^2_+ = \mathbb{R}$ is equipped with a doubling measure μ singular with respect to the Lebesgue measure on $\mathbb{R}$. Finally, the present paper provides a generalization of the celebrated Caffarelli-Sylvestre extension operator from its classical setting of $\mathbb{R}^{n+1}_+$ to general open sets, and hence, an extension of the concept of fractional Laplacian to Ahlfors regular boundaries and beyond.

***Résumé.* (Théorie elliptique dans des domaines à frontières de dimension mixte)** — Soit $\Omega \subset \mathbb{R}^n$ un domaine dont la frontière peut contenir des morceaux de dimensions différentes. Par exemple, Ω peut être une boule de $\mathbb{R}^3$, moins l'un de ses diamètres D, ou un domaine en dents de scies, avec une frontière composée de morceaux de courbes et de morceaux de sphères. Ou encore, un domaine avec une frontière (partiellement) fractale. Avec des hypothèses géométriques convenables, essentiellement l'existence de mesures doublantes sur Ω et $\partial\Omega$ de tailles appropriées, on construit une classe d'opérateurs elliptiques d'ordre 2 dégénérés de manière adaptée à la géométrie, et on

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prouve les estimations clé associées à ces opérateurs L. Ceci inclue des inégalités de Poincaré et de Harnack, le principe du maximum, et la continuité Höldérienne à la frontière des solutions. On introduit les espaces de Hilbert naturellement associés à la géométrie, on construit les opérateurs de trace et d'extension associés, on les utilise pour définir les solutions faibles de $Lu = 0$, puis on prouve les inégalités de De Giorgi-Nash-Moser dans Ω et à la frontière, on résout le problème de Dirichlet, qu'on utilise pour construire une mesure elliptique ω_L associée à L. On construit les fonctions de Green et on les utilise pour obtenir le principe de comparaison et la propriété doublante pour ω_L. Notre théorie étant centrée sur les mesure, en pas seulement sur la geometrie de Ω, les résultats sont nouveaux même dans le cas classique du demi-plan $\mathbb{R}^2_+$, mais où la frontière $\partial\mathbb{R}^2_+ = \mathbb{R}$ est munie d'une mesure doublante μ singulière par rapport à la mesure de Lebesgue sur $\mathbb{R}$. Finalement, ce papier donne une généralisation du célèbre opérateur d'extension de Caffarelli-Sylvestre, depuis son cadre classique de $\mathbb{R}^{n+1}_+$ vers des ouverts plus généraux, et donc une extension du concept de Laplacien fractionnaire à des frontières Ahlfors regulières et au delà.

CONTENTS

CHAPTER 1

MOTIVATION
AND
A GENERAL OVERVIEW OF THE MAIN RESULTS

1.1. Motivation

Massive efforts of the past few decades at the intersection of analysis, PDEs, and geometric measure theory have recently culminated in a comprehensive understanding of the relationship between the absolute continuity of the harmonic measure with respect to the Hausdorff measure and rectifiability of the underlying set [**4, 5**]. Even more recently, in 2020, we could identify a sharp class of elliptic operator for which the elliptic measure behaves similarly to that of the Laplacian in the sense that analogues of the above results could be obtained, at least under mild additional topological assumptions [**40, 46**].

Unfortunately, all of those results have been restricted to the case of n-dimensional domains with $n-1$ dimensional boundaries, and as such, left completely beyond the scope of the discussion a higher co-dimensional case, such as, for example, a complement of a curve in $\mathbb{R}^3$. The authors of the present paper have recently launched a program investigating the latter, which we will partially review below, and which quite curiously brought a completely different level of understanding of $n-1$ dimensional results and a plethora of open problems, again, relevant even in the context of "classical" geometries, e.g., simply connected planar domains or even a half-space. What is the role of measure on the boundary and given a rough measure, possibly singular with respect to the Hausdorff measure, can we define an elliptic operator whose solutions would be well-behaved near the boundary? What is the role of the dimension, especially when fractional dimensions are allowed? Even in the case of the Laplacian the dimension of the harmonic measure is a mysterious and notoriously difficult subject with scarce celebrated results due to Makarov, Bourgain, Wolff, and many problems open to this date, but what if we step out of the context of the Laplacian and similar operators? Closely related to this question is another one: what is the role of degeneracy, that is, where are the limits of the concept of "ellipticity" which could still carry reasonable PDE properties. This brought us, in particular, to a new version of the Caffarelli-Sylvestre extension operator and hence, a new fractional Laplacian (or, one could say, a certain form of differentiation) on general Ahlfors regular sets. Let us discuss all this in more details.

As we mentioned above, this project started as a continuation of efforts in [**15, 18, 16, 17, 14, 51, 30**] to define an analogue of harmonic measure on domains with lower dimensional boundaries and ultimately to develop a PDE theory comparable in power and scope to that of $n-1$-rectifiable sets. Initially, we focused on domains $\Omega \subset \mathbb{R}^n$ whose boundary $\Gamma = \partial\Omega$ is Ahlfors regular of dimension $d < n-1$ (see (2.1) below). When $d \leq n-2$, such sets would not be recognized by harmonic functions, and we were led to a class of degenerate elliptic operators L adapted to the dimension. Taking the coefficients of L to be, roughly speaking, of the order of $\operatorname{dist}(x, \partial\Omega)^{-(n-d-1)}$, we managed to define a well behaved elliptic measure ω_L associated to L and Ω and prove the estimates for ω_L and for the Green functions, similar to the classical situation where $d = n-1$ and L is elliptic. Furthermore, we proved in [**16**] that ω_L is absolutely continuous with respect to the Hausdorff measure $\mu = \mathscr{H}^d_{|\Gamma}$, with an A_∞ density, when Γ is a Lipschitz graph with a small Lipschitz constant and the coefficients of L are proportional to $D(x, \partial\Omega)^{-(n-d-1)}$, where D is a carefully chosen, appropriately smooth, distance function. However, in an effort to extend these results to the context of uniformly rectifiable domains we faced some fundamental problems which bring us to the setting of the present paper.

A key feature of (uniformly) rectifiable sets is the fact that at every scale a significant portion of such a set can be suitably covered by well-controlled Lipschitz images. To take advantage of this, one has to develop an intricate procedure which allows one to "hide the bad parts" and more precisely, it is absolutely essential to be able to consider suitable subdomains of an initial domain which carry similar estimates on harmonic measure, within the scales under consideration. The latter are referred to as the saw-tooth domains and the reader can imagine "biting off" from the initial domain a ball, or rather a cone, surrounding a bad subset of the boundary. The problem is that when the initial domain is, say, the complement of a curve in $\mathbb{R}^3$, any subdomain would have a boundary of a mixed dimension and the specific procedure that we are describing yields pieces of one-dimensional curves intercepted by 2-dimensional spheres, or more precisely, 2-dimensional Lipschitz images. We will give in Section 3 a careful description of this example. Similarly, any attempt to localize a problem on a set with lower dimensional boundary (e.g., $\mathbb{R}^n \setminus \mathbb{R}^d$) yields a new domain, given by an n dimensional ball minus a d-dimensional curve, which now has a union of an $n-1$ dimensional sphere and a d-dimensional surface as its boundary. These challenges led us to a necessity to develop a meaningful elliptic theory in the presence of the mixed-dimensional boundaries.

This immediately raises a question: what are the appropriate elliptic operators, as our favorite choice $L = -\operatorname{div} D(x, \partial\Omega)^{-(n-d-1)}\nabla$, and similar ones, carry a power which depends on the dimension of the boundary d. To some extent, this is necessary: as we mentioned above, the Laplacian would not see very low-dimensional sets and this argument can be generalized. But to which extent? Can $L = -\operatorname{div} D(x, \partial\Omega)^{-(n-d-1+\beta)}\nabla$ be allowed for some β? Can