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TRANSSERIAL HARDY FIELDS

by

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Abstract. — It is well known that Hardy fields can be extended with integrals, exponentials and solutions to Pfaffian first order differential equations $f' = P(f)/Q(f)$. From the formal point of view, the theory of transseries allows for the resolution of more general algebraic differential equations. However, until now, this theory did not admit a satisfactory analytic counterpart. In this paper, we will introduce the notion of a transserial Hardy field. Such fields combine the advantages of Hardy fields and transseries. In particular, we will prove that the field of differentially algebraic transseries over $\mathbb{R}\{\{x^{-1}\}\}$ carries a transserial Hardy field structure. Inversely, we will give a sufficient condition for the existence of a transserial Hardy field structure on a given Hardy field.

***Résumé* (Corps de Hardy transsériels).** — Il est bien connu que des corps de Hardy peuvent être étendus par des intégrales, des exponentielles et des solutions d'équations différentielles Pfaffiennes du type $f' = P(f)/Q(f)$. D'un point de vue formel, la théorie des transséries permet la résolution d'équations différentielles algébriques plus générales. Toutefois, cette théorie n'admettait pas encore de contre-partie analytique satisfaisante jusqu'à présent. Dans cet article, nous introduisons la notion de corps de transséries transsériel. Ces corps combinent les avantages des corps de Hardy et de la théorie des transséries. En particulier, nous démontrons que le corps des transséries vérifiant une équation différentiello-algébrique sur $\mathbb{R}\{\{x^{-1}\}\}$ possède une structure de corps de Hardy transsériel. Réciproquement, nous donnerons une condition suffisante pour l'existence d'une structure transsérielle sur un corps de Hardy donné.

1. Introduction

A Hardy field is a field of infinitely differentiable germs of real functions near infinity. Since any non-zero element in a Hardy field $\mathscr{H}$ is invertible, it admits no zeros in a suitable neighbourhood of infinity, whence its sign remains constant. It follows that Hardy fields both carry a total ordering and a valuation. The ordering and valuation can be shown to satisfy several natural compatibility axioms with the

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differentiation, so that Hardy fields are models of the so called theory of H-fields [**1, 3, 2**].

Other natural models of the theory of H-fields are fields of transseries [**23, 31, 15, 16, 27, 26**]. Contrary to Hardy fields, these models are purely formal, which makes them particularly useful for the automation of asymptotic calculus [**23**]. Furthermore, the so called field of grid-based transseries $\mathbb{T}$ (for instance) satisfies several remarkable closure properties. Namely, $\mathbb{T}$ is differentially Henselian [**26**, theorem 8.21] and it satisfies the differential intermediate value theorem [**26**, theorem 9.33].

Now the purely formal nature of the theory of transseries is also a drawback, since it is not *a priori* clear how to associate a genuine real function to a transseries f, even in the case when f satisfies an algebraic differential equation over $\mathbb{R}\{\{x^{-1}\}\}$. One approach to this problem is to develop Écalle's accelero-summation theory [**17, 18, 19, 20, 11, 12**], which constitutes a more or less canonical way to associate analytic functions to formal transseries with a "natural origin". In this paper, we will introduce another approach, based on the concept of a *transserial Hardy field*.

Roughly speaking, a transserial Hardy field is a truncation-closed differential subfield $\mathscr{T}$ of $\mathbb{T}$, which is also a Hardy field. The main objectives of this paper are to show the following two things:

1. The differentially algebraic closure in $\mathbb{T}$ of a transserial Hardy field can be given the structure of a transserial Hardy field.
2. Any differentially algebraic Hardy field extension of a transserial Hardy field, which is both differentially Henselian and closed under exponentiation, admits a transserial Hardy field structure.

We have chosen to limit ourselves to the context of grid-based transseries. More generally, an interesting question is which H-fields can be embedded in fields of well-based transseries and which differential fields of well-based transseries admit Hardy field representations. We hope that work in progress [**5, 4**] on the model theory of H-fields and asymptotic fields will enable us to answer these questions in the future.

The theory of Hardy fields admits a long history. Hardy himself proved that the field of so called L-functions is a Hardy field [**21, 22**]. The definition of a Hardy field and the possibility to add integrals, exponentials and algebraic functions is due to Bourbaki [**10**]. More generally, Hardy fields can be extended by the solutions to Pfaffian first order differential equations [**32, 6**] and solutions to certain second order differential equations [**9**]. Further results on Hardy fields can be found in [**28, 29, 30, 7, 8**]. The theory of transserial Hardy fields can be thought of as a systematic way to deal with differentially algebraic extensions of any order.

The main idea behind the addition of solutions to higher order differential equations to a given transserial Hardy field $\mathscr{T}$ is to write such solutions in the form of "integral series" over $\mathscr{T}$ (see also [**25**]). For instance, consider a differential equations such as

$$f' = \mathrm{e}^{-2\mathrm{e}^x} + f^2,$$

for large $x \succ 1$. Such an equation may typically be written in integral form

$$f = \int \mathrm{e}^{-2\mathrm{e}^x} + \int f^2.$$

The recursive replacement of the left-hand side by the right-hand side then yields a "convergent" expansion for f using iterated integrals

$$f = \int \mathrm{e}^{-2\mathrm{e}^x} + \int \left(\int \mathrm{e}^{-2\mathrm{e}^x}\right)^2 + 2\left(\int \mathrm{e}^{-2\mathrm{e}^x}\right)\left(\int \left(\int \mathrm{e}^{-2\mathrm{e}^x}\right)^2\right) + \cdots,$$

where we understand that each of the integrals in this expansion are taken from $+\infty$:

$$\left(\int g\right)(x) = \int_{\infty}^{x} g(t)\mathrm{d}t.$$

In order to make this idea work, one has to make sure that the extension of $\mathcal{T}$ with a solution f of the above kind does not introduce any oscillatory behaviour. This is done using a combination of arguments from model theory and differential algebra.

More precisely, whenever a transseries solution f to an algebraic differential equation over $\mathcal{T}$ is not yet in $\mathcal{T}$, then we may assume the equation to be of minimal "complexity" (a notion which refines Ritt rank). In section 2, we will show how to put the equation in normal form

$$Lf = P(f), \tag{1}$$

where $P \in \mathcal{T}\{F\}$ is "small" and $L \in \mathcal{T}[\partial]$ admits a factorization

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

over $\mathcal{T}[\mathrm{i}]$. In section 4, it will be show how to solve (1) using iterated integrals, using the fact that the equation $(\partial - \varphi)f = g$ admits $\mathrm{e}^{\int \varphi} \int \mathrm{e}^{-\int \varphi} g$ as a solution. Special care will be taken to ensure that the constructed solution is again real and that the solution admits the same asymptotic expansion over $\mathcal{T}$ as the formal solution.

Section 3 contains some general results about transserial Hardy fields. In particular, we prove the basic extension lemma: given a transseries f and a real germ $\hat{f}$ at infinity which behave similarly over $\mathcal{T}$ (both from the asymptotic and differentially algebraic points of view), there exists a transserial Hardy field extension of $\mathcal{T}$ in which f and $\hat{f}$ may be identified. The differential equivalence of f and $\hat{f}$ will be ensured by the fact that the equation (1) was chosen to be of minimal complexity. Using Zorn's lemma, it will finally be possible to close $\mathcal{T}$ under the resolution of real differentially algebraic equations. This will be the object of the last section 5. Throughout the paper, we will freely use notations from **[26]**. For the reader's convenience, some of the notations are recalled in section 2.1. We also included a glossary at the end.

It would be interesting to investigate whether the theory of transserial Hardy fields can be generalized so as to model some of the additional compositional structure on $\mathbb{T}$. A first step would be to replace all differential polynomials by restricted analytic functions [**14**]. A second step would be to consider postcompositions with operators $x+\delta$ for sufficiently flat transseries f for which Taylor's formula holds:

$$f\circ(x+\delta)=f+f'\delta+\frac{1}{2}f''\delta^2+\cdots.$$

This requires the existence of suitable analytic continuations of f in the complex domain. Typically, if $f\in\mathbb{T}_{\preceq\!\!\!\prec g}$ with $g\in\mathbb{T}^{>,\succ}$, then $f\circ g^{\mathrm{inv}}$ should be defined on some sector at infinity (notice that this can be forced for the constructions in this paper). Finally, more violent difference equations, such as

$$f(x)=\frac{1}{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}}+f(x+1),$$

generally give rise to quasi-analytic solutions. From the model theoretic point view, they can probably always be seen as convergent sums.

Finally, one may wonder about the respective merits of the theory of accelero-summation and the theory of transserial Hardy fields. Without doubt, the first theory is more canonical and therefore has a better behaviour with respect to composition. In particular, we expect it to be easier to prove o-minimality results [**13**]. On the other hand, many technical details still have to be worked out in full detail. This will require a certain effort, even though the resulting theory can be expected to have many other interesting applications. The advantage of the theory of transserial Hardy fields is that it is more direct (given the current state of art) and that it allows for the association of Hardy field elements to transseries which are not necessarily accelero-summable.

2. Preliminaries

2.1. Notations. — Let $\mathbb{T}=\mathbb{R}[\![x]\!]=\mathbb{R}[\![\mathfrak{T}]\!]$ be the totally ordered field of grid-based transseries, as in [**26**]. Any transseries is an infinite linear combination $f=\sum_{\mathfrak{m}\in\mathfrak{T}}f_{\mathfrak{m}}\mathfrak{m}$ of transmonomials, with grid-based support $\operatorname{supp} f\subseteq\mathfrak{T}$. Transmonomials $\mathfrak{m},\mathfrak{n},\ldots$ are systematically written using the fraktur font. Each transmonomial is an iterated logarithm $\log_l x$ of x or the exponential of a transseries g with $\mathfrak{n}\succ 1$ for each $\mathfrak{n}\in\operatorname{supp} g$. The asymptotic relations $\preccurlyeq,\prec,\asymp,\sim,\preceq\!\!\!\prec,\prec\!\!\!\prec,\asymp\!\!\!\asymp$ and $\approx$ on $\mathbb{T}$ are defined by

$$\begin{aligned}
f\preccurlyeq g &\iff f=O(g)\\
f\prec g &\iff f=o(g)\\
f\asymp g &\iff f\preccurlyeq g\preccurlyeq f\\
f\sim g &\iff f-g\prec g\\
f\preceq\!\!\!\prec g &\iff \log|f|\preccurlyeq\log|g|\\
f\prec\!\!\!\prec g &\iff \log|f|\prec\log|g|\\
f\asymp\!\!\!\asymp g &\iff \log|f|\asymp\log|g|\\
f\approx g &\iff \log|f|\sim\log|g|.
\end{aligned}$$

Given $\mathfrak{v} \neq 1$, one also defines variants of $\preccurlyeq, \prec$, etc. modulo flatness:

$$\begin{array}{rcl} f \preccurlyeq_{\mathfrak{v}} g & \Longleftrightarrow & \exists \mathfrak{m} \prec\!\!\!\prec \mathfrak{v}, f \preccurlyeq g\mathfrak{m} \\ f \prec_{\mathfrak{v}} g & \Longleftrightarrow & \forall \mathfrak{m} \prec\!\!\!\prec \mathfrak{v}, f \prec g\mathfrak{m} \\ f \preccurlyeq^*_{\mathfrak{v}} g & \Longleftrightarrow & \exists \mathfrak{m} \preccurlyeq\!\!\!\preccurlyeq \mathfrak{v}, f \preccurlyeq g\mathfrak{m} \\ f \prec^*_{\mathfrak{v}} g & \Longleftrightarrow & \forall \mathfrak{m} \preccurlyeq\!\!\!\preccurlyeq \mathfrak{v}, f \prec g\mathfrak{m}. \end{array}$$

It is convenient to use relations as superscripts in order to filter elements, as in

$$\begin{array}{rcl} \mathbb{T}^{>} & = & \{f \in \mathbb{T} : f > 0\} \\ \mathbb{T}^{\neq} & = & \{f \in \mathbb{T} : f \neq 0\} \\ \mathbb{T}^{\succ} & = & \{f \in \mathbb{T} : f \succ 1\}. \end{array}$$

Similarly, we use subscripts for filtering on the support:

$$\begin{array}{rcl} f_{\succ} & = & \displaystyle\sum_{\mathfrak{m} \in \operatorname{supp} f, \mathfrak{m} \succ 1} f_{\mathfrak{m}} \mathfrak{m} \\ f_{\prec\!\!\!\prec \mathfrak{v}} & = & \displaystyle\sum_{\mathfrak{m} \in \operatorname{supp} f, \mathfrak{m} \preccurlyeq\!\!\!\preccurlyeq \mathfrak{v}} f_{\mathfrak{m}} \mathfrak{m} \\ \mathbb{T}_{\succ} & = & \{f_{\succ} : f \in \mathbb{T}\} \\ \mathbb{T}_{\prec\!\!\!\prec \mathfrak{v}} & = & \{f_{\prec\!\!\!\prec \mathfrak{v}} : f \in \mathbb{T}\}. \end{array}$$

We denote the derivation on $\mathbb{T}$ w.r.t. x by ∂ and the corresponding distinguished integration (with constant part zero) by $\int$. The logarithmic derivative of f is denoted by $f^{\dagger}$. The operations $\uparrow$ and $\downarrow$ of upward and downward shifting correspond to post-composition with $\exp x$ resp. $\log x$. We finally write $f \trianglelefteq g$ if the transseries f is a truncation of g, i.e. $\mathfrak{m} \prec \operatorname{supp} f$ for all $\mathfrak{m} \in \operatorname{supp}(g - f)$.

2.2. Differential fields of transseries and cuts. — Given $f \in \mathbb{T}$, we define the *canonical span* of f by

$$\operatorname{span} f = \max_{\preccurlyeq\!\!\!\preccurlyeq} \{\mathrm{e}^{-\partial(\log(\mathfrak{m}/\mathfrak{n}))} : \mathfrak{m}, \mathfrak{n} \in \operatorname{supp} f\}. \tag{2}$$

By convention, $\operatorname{span} f = 1$ if $\operatorname{supp} f$ contains less than two elements. We also define the *ultimate canonical span* of f by

$$\operatorname{uspan} f = \min_{\preccurlyeq\!\!\!\preccurlyeq} \{\operatorname{span} f_{\prec \mathfrak{v}} : \mathfrak{v} \in \operatorname{supp} f\}. \tag{3}$$

We notice that $\operatorname{uspan} f \neq 1$ if and only if $\operatorname{supp} f$ admits no minimal element for $\preccurlyeq$.

Example 1. — We have

$$\begin{array}{rcl} \operatorname{span}\left(1 + \dfrac{\mathrm{e}^{-x}}{1 - x^{-1}}\right) & = & \mathrm{e}^{-x} \\ \operatorname{uspan}\left(1 + \dfrac{\mathrm{e}^{-x}}{1 - x^{-1}}\right) & = & x^{-1} \end{array}$$