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ON THE STRUCTURE OF TRIANGULATED CATEGORIES WITH FINITELY MANY INDECOMPOSABLES

by Claire Amiot

Abstract. — We study the problem of classifying triangulated categories with finite-dimensional morphism spaces and finitely many indecomposables over an algebraically closed field k. We obtain a new proof of the following result due to Xiao and Zhu: the Auslander-Reiten quiver of such a category $\mathcal{T}$ is of the form $\mathbb{Z}\Delta/G$ where Δ is a disjoint union of simply-laced Dynkin diagrams and G a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. Then we prove that for 'most' groups G, the category $\mathcal{T}$ is standard, *i.e.* k-linearly equivalent to an orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$. This happens in particular when $\mathcal{T}$ is maximal d-Calabi-Yau with $d \geq 2$. Moreover, if $\mathcal{T}$ is standard and algebraic, we can even construct a triangle equivalence between $\mathcal{T}$ and the corresponding orbit category. Finally we give a sufficient condition for the category of projectives of a Frobenius category to be triangulated. This allows us to construct non standard 1-Calabi-Yau categories using deformed preprojective algebras of generalized Dynkin type.

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Résumé (*Sur la structure des catégories triangulées*). — Cet article traite du problème de classification des catégories triangulées sur un corps algébriquement clos k dont les espaces de morphismes sont de dimension finie et avec un nombre fini d'indécomposables. Nous obtenons une nouvelle preuve du résultat suivant dû à Xiao et Zhu : le carquois d'Auslander-Reiten d'une telle catégorie $\mathcal{T}$ est de la forme $\mathbb{Z}\Delta/G$ où Δ est une union disjointe de diagrammes de Dynkin simplement lacés et G est un groupe d'automorphismes de $\mathbb{Z}\Delta$ faiblement admissible. Nous montrons ensuite que pour 'presque' tous groupes G, la catégorie $\mathcal{T}$ est standard, c'est-à-dire k-linéairement équivalente à une catégorie d'orbites $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$. C'est en particulier le cas lorsque $\mathcal{T}$ est maximale d-Calabi-Yau avec $d \geq 2$. De plus, si $\mathcal{T}$ est standard et algébrique, nous pouvons même construire une équivalence triangulée entre $\mathcal{T}$ et la catégorie d'orbites correspondante. Nous donnons finalemant une condition suffisante pour que la catégorie de projectifs d'une catégorie de Frobenius soit triangulée. Cela nous permet de construire des catégories 1-Calabi-Yau non standard en utilisant les algèbres préprojectives déformées de type Dynkin généralisé.

Introduction

Let k be an algebraically closed field and $\mathcal{T}$ a small Krull-Remak-Schmidt k-linear triangulated category (see [47]). We assume that

a) $\mathcal{T}$ is Hom-*finite*, *i.e.* the space $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is finite-dimensional for all objects X, Y of $\mathcal{T}$.

It follows that indecomposable objects of $\mathcal{T}$ have local endomorphism rings and that each object of $\mathcal{T}$ decomposes into a finite direct sum of indecomposables [17, 3.3]. We assume moreover that

b) $\mathcal{T}$ is *locally finite*, *i.e.* for each indecomposable X of $\mathcal{T}$, there are at most finitely many isoclasses of indecomposables Y such that $\mathrm{Hom}_{\mathcal{T}}(X,Y) \neq 0$.

It was shown in [48] that condition *b*) implies its dual. Condition *b*) holds in particular if we have

b′) $\mathcal{T}$ is *additively finite*, *i.e.* there are only finitely many isomorphism classes of indecomposables in $\mathcal{T}$.

The study of particular classes of such triangulated categories $\mathcal{T}$ has a long history. Let us briefly recall some of its highlights:

1) If A is a representation-finite selfinjective algebra, then the stable category $\mathcal{T}$ of finite-dimensional (right) A-modules satisfies our assumptions and is additively finite. The structure of the underlying k-linear category of $\mathcal{T}$ was determined by C. Riedtmann in [39], [40], [41] and [42].

2) In [21], D. Happel showed that the bounded derived category of the category of finite-dimensional representations of a representation-finite quiver is locally finite and described its underlying k-linear category.

3) The stable category $\underline{\mathrm{CM}}(R)$ of Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity R of dimension d is a Hom-finite triangulated category which is $(d-1)$-Calabi-Yau (cf. for example [28] and [50]). In [4], M. Auslander and I. Reiten showed that if the dimension of R is 1, then the category $\underline{\mathrm{CM}}(R)$ is additively finite and computed the shape of the components of its Auslander-Reiten quiver.

4) The cluster category $\mathcal{C}_Q$ of a finite quiver Q without oriented cycles was introduced in [12] if Q is an orientation of a Dynkin diagram of type $\mathbb{A}$ and in [11] in the general case. The category $\mathcal{C}_Q$ is triangulated [30] and, if Q is representation-finite, satisfies a) and b$'$).

In a recent article [48], J. Xiao and B. Zhu determined the structure of the Auslander-Reiten quiver of a locally finite triangulated category. In this paper, we obtain the same result with a new proof in Section 4, namely that each connected component of the Auslander-Reiten quiver of the category $\mathcal{T}$ is of the form $\mathbb{Z}\Delta/G$, where Δ is a simply-laced Dynkin diagram and G is trivial or a weakly admissible group of automorphisms. Contrary to J. Xiao and B. Zhu, we do not discuss separately the case where the Auslander-Reiten contains a loop.

We are interested in the k-linear structure of $\mathcal{T}$. If the Auslander-Reiten quiver of $\mathcal{T}$ is of the form $\mathbb{Z}\Delta$, we show that the category $\mathcal{T}$ is standard, *i.e.* it is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. Then in Section 6, we prove that $\mathcal{T}$ is standard if the number of vertices of $\Gamma = \mathbb{Z}\Delta/G$ is strictly greater than the number of isoclasses of indecomposables of $\operatorname{mod} k\Delta$. In the last section, using [8] we construct examples of non standard triangulated categories such that $\Gamma = \mathbb{Z}\Delta/\tau$.

Finally, in the standard cases, we are interested in the triangulated structure of $\mathcal{T}$. For this, we need to make additional assumptions on $\mathcal{T}$. If the Auslander-Reiten quiver is of the form $\mathbb{Z}\Delta$, and if $\mathcal{T}$ is the base of a tower of triangulated categories [29], we show that there is a triangle equivalence between $\mathcal{T}$ and the derived category $\mathcal{D}^b(\operatorname{mod} k\Delta)$. For the additively finite cases, we have to assume that $\mathcal{T}$ is standard and algebraic in the sense of [31]. We then show that $\mathcal{T}$ is (algebraically) triangle equivalent to the orbit category of $\mathcal{D}^b(\operatorname{mod} k\Delta)$ under the action of a weakly admissible group of automorphisms. In particular, for each $d \geq 2$, the algebraic triangulated categories with finitely many indecomposables which are maximal Calabi-Yau of CY-dimension d are parametrized by the simply-laced Dynkin diagrams.

Our results apply in particular to many stable categories $\underline{\operatorname{mod}}\, A$ of representation-finite selfinjective algebras A. These algebras were classified up to stable equivalence by C. Riedtmann [40], [42] and H. Asashiba [1]. In [9], J. Białkowski and A. Skowroński give a necessary and sufficient condition on these algebras so that their stable categories $\underline{\mathrm{mod}}\,A$ are Calabi-Yau. In [26] and [27], T. Holm and P. Jørgensen prove that certain stable categories $\underline{\mathrm{mod}}\,A$ are in fact d-cluster categories. These results can also be proved using our Corollary 7.3.

This paper is organized as follows: In Section 1, we prove that $\mathcal{T}$ has Auslander-Reiten triangles. Section 2 is dedicated to definitions about stable valued translation quivers and admissible automorphisms groups [23], [24], [14]. We show in Section 3 that the Auslander-Reiten quiver of $\mathcal{T}$ is a stable valued quiver and in Section 4, we reprove the result of J. Xiao and B. Zhu [48]: The Auslander-Reiten quiver is a disjoint union of quivers $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$, and G a weakly admissible group of automorphisms. In Section 5, we construct a covering functor $\mathcal{D}^b(\mathrm{mod}\,k\Delta) \to \mathcal{T}$ using Riedtmann's method [39]. Then, in Section 6, we exhibit some combinatorial cases in which $\mathcal{T}$ has to be standard, in particular when $\mathcal{T}$ is maximal d-Calabi-Yau with $d \geq 2$. Section 7 is dedicated to the algebraic case. If $\mathcal{T}$ is algebraic and standard, we can construct a triangle equivalence between $\mathcal{T}$ and an orbit category. If $\mathcal{P}$ is a k-category such that $\mathrm{mod}\,\mathcal{P}$ is a Frobenius category satisfying certain conditions, we will prove in Section 8 that $\mathcal{P}$ has naturally a triangulated structure. This allows us to deduce in Section 9 that the category $\mathrm{proj}\,P^f(\Delta)$ of the projective modules over a deformed preprojective algebra of generalized Dynkin type [8] is naturally triangulated and to reduce the classification of the additively finite triangulated categories which are 1-Calabi-Yau to that of the deformed preprojective algebras in the sense of [8]. In particular, thanks to [8], we obtain the existence of non standard 1-Calabi-Yau categories in characteritic 2. Using our results and an extension of those of [8], Białkowski and Skowroński have recently proved [10] the existence of non standard 1-Calabi-Yau categories in characteristic 3. This is noteworthy since in characteristic different from 2, additively finite *module categories* are always standard [6].

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Notation and terminology

We work over an algebraically closed field k. By a *triangulated category*, we mean a k-linear triangulated category $\mathcal{T}$. We write S for the suspension functor of $\mathcal{T}$ and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ for a distinguished triangle. We say that $\mathcal{T}$ is Hom-*finite* if for each pair X, Y of objects in $\mathcal{T}$, the space