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# POINCARÉ'S PROOF OF THE SO-CALLED BIRKHOFF-WITT THEOREM

Tuong TON-THAT, Thai-Duong TRAN (*)

*In honor of the 100th birthday of the article, "Sur les groupes continus"*

ABSTRACT. — A methodical analysis of the research related to the article, "Sur les groupes continus", of Henri Poincaré reveals many historical misconceptions and inaccuracies regarding his contribution to Lie theory. A thorough reading of this article confirms the priority of his discovery of many important concepts, especially that of the *universal enveloping algebra* of a Lie algebra over the real or complex field, and the *canonical map* (*symmetrization*) of the symmetric algebra onto the universal enveloping algebra. The essential part of this article consists of a detailed discussion of his rigorous, complete, and enlightening proof of the so-called Birkhoff-Witt theorem.

RÉSUMÉ. — LA DÉMONSTRATION DE POINCARÉ DU THÉORÈME DIT DE BIRKHOFF-WITT. — Une analyse méthodique des travaux faits en connexion avec l'article, "Sur les groupes continus", de Henri Poincaré révèle des erreurs historiques et des jugements injustes en ce qui concerne sa contribution à la théorie de Lie. Une étude approfondie de cet article confirme l'antériorité de sa découverte de plusieurs concepts importants; notamment de *l'algèbre enveloppante universelle* d'une algèbre de Lie sur le corps réel ou le corps complexe, et de *l'application canonique* (*la symétrisation*) de l'algèbre symétrique sur l'algèbre enveloppante universelle. L'essentiel de cet article consiste en un examen approfondi de sa démonstration rigoureuse et complète du théorème de Birkhoff-Witt.

## 1. INTRODUCTION

In our research on the universal enveloping algebras of certain infinite-dimensional Lie algebras we were led to study in detail the original proofs

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Tuong TON-THAT and Thai-Duong TRAN, Department of Mathematics, University of Iowa, Iowa City, IA 52242 (USA).
E-mail: tonthat@math.uiowa.edu and ttran@math.uiowa.edu.

of the so-called Birkhoff-Witt theorem (more recently, Poincaré-Birkhoff-Witt theorem). This, in turn, led us to the investigation of Poincaré's contribution to Lie theory (*i.e.*, the theory of Lie groups, Lie algebras, and their representations). To our great surprise we discovered many historical misconceptions and inaccuracies, even in some of the classics written by the leading authorities on the subject. This discovery has puzzled us for some time, and we have sought the opinions of several experts in the field. Their answers together with our thorough reading of several original articles on the subject shed some light on this mystery. We were astounded to find out that Poincaré was given credit neither for his fundamental discovery of the *universal enveloping algebra* of a Lie algebra over a field of characteristic zero, nor for his introduction of the *symmetrization map*, and only a cursory and belated acknowledgment of his contribution to the so-called *Birkhoff-Witt theorem*, of which he gave a rigorous, complete, beautiful, and enlightening proof. Indeed, in two of the most exhaustive treatises on universal enveloping algebra [Cohn 1981] and [Dixmier 1974], Poincaré [1900] was not mentioned. In many authoritative textbooks treating Lie theory such as [Chevalley 1955], [Cartan & Eilenberg 1956], [Kuros 1963], [Jacobson 1962], [Varadarajan 1984 (1974)], [Humphreys 1972], [Knapp 1986], …, Poincaré's discovery of the universal enveloping algebra and the symmetrization map was ignored. In some books his name was left off the Birkhoff-Witt theorem, and his fundamental article [Poincaré 1900] was not even quoted. In the *Encyclopaedia of Mathematics* [*Encyclopaedia* 1988–1994] under the rubric "Birkhoff-Witt theorem" it was written "…*The first variant of this theorem was obtained by H. Poincaré; the theorem was subsequently completely demonstrated by E. Witt* [1937] *and G.D. Birkhoff*[1] [1937]…". Clearly the author, T.S. Fofanova, did not read carefully [Poincaré 1900]; otherwise she would have realized that Poincaré had discovered and completely demonstrated this theorem at least thirty-seven years before Witt and Birkhoff. Why such slights can happen to one of the greatest mathematicians of all times, who published

[1] Actually Garrett Birkhoff (1911–1996), not G.D. (Birkhoff) which are the initials of George David Birkhoff (1884–1944), the father of Garrett. This inaccuracy only occurs in the translation, not in the original (Russian) version of the *Encyclopaedia*. We are grateful to Professor Sergei Silvestrov for elucidating this fact to us.

these results [Poincaré 1900] in one of the most prestigious scientific journals, *Transactions of the Cambridge Philosphical Society*, on the occasion of the jubilee of another great mathematician, Sir George Gabriel Stokes, is a most interesting mystery that we shall attempt to elucidate in this article. But before beginning our investigation we want to make it clear that our intention is to study thoroughly one of the most fundamental discoveries by one of the greatest minds in order to understand how important ideas are created, and not to rectify such injustices, for such a task is doomed to fail as the force of habit always prevails; a fact very clearly expressed in the following excerpt from [Gittleman 1975, p. 186], "*... l'Hospital's rule, Maclaurin's series, Cramer's rule, Rolle's theorem, and Taylor's series are familiar terms to calculus students. Actually, only one of these five mathematicians was the original discoverer of the result attributed to him, and that man was Rolle. The person who popularizes a result generally has his name attached to it, although later it may be learned that someone else had originally discovered the same result. For practical purposes names are not changed, but even so, the mistakes seem to compensate for one another. Although Maclaurin was credited with a series he did not discover, a rule which he did originate is now known as Cramer's rule...*". Besides, Poincaré is a member of the elite group of mathematicians to whom many important mathematical discoveries are attributed; indeed, in the *Encyclopaedia of Mathematics* [*Encyclopaedia* 1988–1994] 18 rubrics are listed under his name. Curiously, under the heading "*Poincaré last theorem*" the editorial comments state that "[this theorem] *is also known as the Poincaré-Birkhoff fixed-point theorem,*" and the author, M.I. Voĭtsekhovskiĭ, wrote "*...it was proved by him in a series of particular cases but he did not, however, obtain a general proof of this theorem*".[2] Misnaming mistakes seem to compensate one another after all.

[2] Voĭtsekhovskiĭ continues, "*The paper was sent by Poincaré to an Italian journal two weeks before his death, and the author expressed his conviction, in an accompanying letter to the editor, of the validity of the theorem in the general case.*" Indeed, on December 9, 1911, having some presentiments that he might not live long, Poincaré wrote a moving letter to Guccia, director and founder of the journal *Rendiconti del Circolo Matematico di Palermo* (*cf.* [Poincaré *Œuvres*, II, p. LXVII]), asking his opinion regarding what has become known as "Poincaré's Last Geometric Theorem" (see [Barrow-Green 1997, §7.4.2, pp. 169–174], for an English translation of the letter and an excellent discussion of the theorem). Mr. Guccia readily accepted the memoir for publication and it appeared on March 10, 1912, just a few months before Poincaré's

In his book [Bell 1937], E.T. Bell, who called Poincaré "*the Last Universalist*", considered the Last Geometric Theorem as Poincaré's "*unfinished symphony*" and wrote "*... And it may be noted that Poincaré turned his universality to magnificent use in disclosing hitherto unsuspected connections between distant branches of mathematics, for example, between continuous groups and linear algebra*". This is exactly the impression we had when reading his article, "Sur les groupes continus".

## 2. POINCARÉ'S WORK ON LIE GROUPS

To assess Poincaré's contribution to Lie theory in general we use two main sources [Poincaré *Œuvres*] and [Poincaré *in memoriam* 1921] and investigate in depth the references cited therein. We start with the article, "Analyse des travaux scientifiques de Henri Poincaré, faite par lui-même" [3] which was written by Poincaré himself in 1901 [Poincaré *in memoriam* 1921, pp. 3–135] at the request of G. Mittag-Leffler (*cf.* "Au lecteur" [Poincaré *in memoriam* 1921, pp. 1–2]). It is part of vol. 38 of the journal, *Acta Math.*, published in 1921 in memory of Henri Poincaré. (Actually, most of vol. 39 published in 1923 is also devoted to Poincaré's work). In the third part of the above-mentioned article, Section XII (Algèbre) and Section XIII (Groupes Continus) are devoted to his contribution to Lie theory. Actually, we think that because of Poincaré's impetus finite-dimensional continuous groups were eventually called Lie groups. Indeed, Poincaré expressed repeatedly his great admiration for Lie's work in [Poincaré 1899] and [Poincaré 1900] and wrote in *Rapport sur les*

death on July 17, 1912 (Sur un théorème de géométrie, *Rendiconti del Circolo Matematico di Palermo*, 33, pp. 375–407 = [Poincaré *Œuvres*, VII, pp. 499–538]). Ultimately it was G.D. Birkhoff who gave a complete proof of this theorem (Proof of Poincaré's geometric theorem, *Trans. Amer. Math. Soc.*, 14 (1913), pp. 14–22 = *Collected Mathematical Papers I*, pp. 673–681) and of its generalization to $n$ dimensions (Une généralisation à $n$ dimensions du dernier théorème de géométrie de Poincaré, *C. R. Acad. Sci. Paris*, 192 (1931), pp. 196–198 = *Collected Mathematical Papers II*, pp. 395–397).

[3] In [Poincaré *Œuvres*] this article is listed as published by *Acta Math.*, 30 (1913), pp. 90–92. In fact, it never existed as such; the editors of Poincaré's collected works probably found the manuscript of the article among his papers with his annotations regarding the journal and the date of publication but due to World War I it appeared eventually in [Poincaré *in memoriam* 1921]. This remark extends to all discrepancies between the intended and actual dates of publication of many of Poincaré's works in [Poincaré *in memoriam* 1921], for example, *Rapport sur les travaux de M. Cartan.*

*travaux de M. Cartan* [Poincaré *in memoriam* 1921, pp. 137–145]: "*Je commencerai par* les groupes continus et finis, *qui ont été introduits par Lie dans la science; le savant norvégien a fait connaître les principes fondamentaux de la théorie, et il a montré en particulier que la structure de ces groupes dépend d'un certain nombre de constantes qu'il désigne par la lettre c affectée d'un triple indice et entre lesquelles il doit y avoir certaines relations... une des plus importantes applications* des groupes de Lie...". So far as we know this is the first time that the name *Lie groups* was explicitly mentioned.

Poincaré's first encounter with Lie theory probably dated back to his article [Poincaré 1881] and its generalization [Poincaré 1883]. The problem he considered there can be phrased in modern language as follows:

For $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ let $\mathrm{GL}_n(\mathbb{C})$, the general linear group of all $n \times n$ invertible complex matrices, act on $\mathbb{C}^n$ via $(X, g) \mapsto Xg$, $g \in \mathrm{GL}_n(\mathbb{C})$. Let $F(X)$ denote a homogeneous form of degree $m$ (*i.e.*, a homogeneous polynomial of degree $m$ in $n$ variables $(x_1, \ldots, x_n)$), find the subgroup $G$ of $\mathrm{GL}_n(\mathbb{C})$ which preserves the form $F$; *i.e.*, $F(Xg) = F(X)$, for all $g \in G$. Conversely, given a subgroup $G$ of $\mathrm{GL}_n(\mathbb{C})$ find all homogeneous forms that are left invariant by $G$. This is precisely the problem of polynomial invariants (*cf.* [Weyl 1946]).

In [Poincaré 1881] and [Poincaré 1883] he found all cubic ternary (of three variables) and quaternary (of four variables) forms that are preserved by certain Abelian groups (which he called "*faisceau de substitutions*"), and he also extended this result to the non-Abelian case. Conversely, he exhibited explicit groups that preserve quadratic and cubic ternary and quaternary forms. For example, in [Poincaré 1881, pp. 239–241] he found the subgroup of the unipotent group

$$\left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \; ; \; \alpha, \beta, \gamma \in \mathbb{C} \right\}$$

which preserves the quadratic form

$$\begin{aligned} &[\, x_1 \quad x_2 \quad x_3 \,] \begin{bmatrix} A_1 & B_3 & B_2 \\ B_3 & A_2 & B_1 \\ B_2 & B_1 & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &\qquad = A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2 + 2B_1 x_2 x_3 + 2B_2 x_1 x_3 + 2B_3 x_1 x_2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A_i, B_i \in \mathbb{C}\,; \; 1 \le i \le 3. \end{aligned}$$