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Many problems in analysis have natural formulations as questions of continuity of linear operators defined on spaces of functions or distributions. Such questions can often be answered by relatively straightforward techniques if they can first be reduced to the study of the operator on an appropriate class of simple elements which, in some convenient sense, generate the entire space. For example, a linear operator mapping the Lebesgue space L^1 into a Banach space is continuous if and only if it is bounded on characteristic functions. The selection of an orthonormal basis which diagonalizes or nearly diagonalizes an operator on the Lebesgue space L^2 offers us a similar type of approach.

The main these of the two papers in this volume is a description of a decomposition into simple building blocks of elements in generalizations of the classical Hardy spaces. In contrast to the Lebesgue spaces, these generalized Hardy spaces are not rearrangement invariant. Rather, they consist of functions (or distributions) which satisfy both size and cancellation conditions. In the case of holomorphic or harmonic functions the cancellation is implicit in the differential equation which the functions satisfy. In the case of the "real variable" theory, the cancellation properties are different, but are still sufficient to allow estimates substantially more subtle than those based solely on size considerations.

The history of the classical Hardy spaces and their modern generalizations is rich and we will not summarize it here (the reader interested in this history could start with the book of Stein and Weiss [10], the survey article by Coifman and Weiss [3], the survey talks in [1] or the Proceedings of the 1978 AMS Summer Institute [11]). Much of the interest in Hardy spaces arose from the observation that they provide a useful substitute for the Lebesgue space L^1. Indeed, many naturally occurring operators in harmonic analysis and in the theory of differential equations which are not bounded on the Lebesgue space L^1 are bounded on the Hardy space H^1. The following is an example of such an operator. Let us start with a function $f(x)$ defined on the real axis in the complex plane. Let $F(z)$ be its holomorphic projection to the upper half plane given by the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt$$

with $z = x + iy$, $y > 0$. Now consider the boundary values $g(x) = \lim_{y \to 0+} F(x + iy)$. The mapping from f to $g \equiv Cf$ is a continuous linear operator from L^p to L^p for $1 < p < \infty$. While this mapping is not continuous at the end points $p = 1$ and $p = \infty$, it is a continuous map on H^1, which is a large subspace of L^1. In fact, H^1 was first defined to be the subspace of L^1 on which the operator C is bounded. It turns out that many other operators, apparently quite different from C, are also bounded on H^1 (but not on L^1).

A breakthrough in the understanding of this space H^1 and its important generalizations to n dimensions was made by C. Feffermen and E.M. Stein in [4]. There, they present various descriptions of these spaces and their duals. Using their ideas it is possible to describe H^1 in terms of basic building blocks called atoms:

Definition. A function $a(x)$ (defined on $\mathbb{R}^n$) is called an atom (actually, a 1-atom) if its support is contained in a ball B, $\|a\|_\infty \leq 1/|B|$ ($|B|$ is the Lebesgue measure of B) and $\int_{\mathbb{R}^n} a(x)dx = 0$.

We can now define the Hardy spaces in terms of these building blocks:

Definition. A function f (defined on $\mathbb{R}^n$) belongs to the Hardy space $H^1(\mathbb{R}^n)$ if there is a sequence of numbers λ_j satisfying $\sum_1^\infty |\lambda_j| < \infty$ and a sequence of atoms a_j so that $f = \sum_1^\infty \lambda_j a_j$. The H^1 norm of f is defined to be the infimum of the expressions $\sum|\lambda_j|$ with respect to all possible representations of f of the type just described.

It is a theorem (see [6]) that this "atomic" space is, indeed, the classical space $H^1(\mathbb{R}^n)$.

The atomic theory of Hardy spaces on $\mathbb{R}^n$ is presented in [3] and [6].

Several comments are in order. First, since $\int|a| \leq 1$ for any atom a, the space $H^1(\mathbb{R}^n)$ is a subspace of L^1. In fact, it is a subspace of L^1 consisting of functions which have a certain subtle type of cancellation. If the condition that atoms have mean value zero were dropped from the definition then the resulting space would be $L^1(\mathbb{R}^n)$. Secondly, for many purposes, the atoms are the natural elementary building blocks on which to analyze various operators. Any linear or sublinear estimate that is obtained for atoms (often by means of a simple argument) extends to all of H^1. For example, it is relatively straightforward to show that pseudodifferential operators (of order zero) map atoms into $L^1(\mathbb{R}^n)$. Hence, such operators map H^1 into L^1. Since such operators are also bounded from L^2 to L^2 one can then use the theory of interpolation of operators to conclude that these operators are also bounded from L^p to L^p for $1 < p < 2$ (that is, one can interpolate between H^1 and L^2 and obtain L^p. This is another important sense in which H^1 is a suitable and natural substitute for L^1). Finally, as was mentioned above, the spaces $H^1(\mathbb{R}^n)$ were originally defined by the requirement that operators similar to the operator C described earlier be bounded. If the spaces are defined that way, then what we have offered as a definition is, in fact, one of the deepest results of the subject.

We have just defined $H^1(\mathbb{R}^n)$ as the space of scalar combinations of basic building blocks that are localized and satisfy a size condition as well as a cancellation condition. This type of approach can be extended to a very large range of other contexts (see [3], [8], [2], [7]). There are situations, however, in which the restriction to functions supported on balls is unnatural or inconvenient. Two such situations are presentsd in the two papers in this volume. In one, atoms are a bit too simple to use for proving that certain operators map a Hardy space into itself; in the other, the building blocks are holomorphic (or harmonic) functions and, thus, cannot have support in balls.

Let us begin by describing the first situation. It is fairly clear that, if $a(x)$ is an atom on $\mathbb{R}^1$, then $Ca \in L^1(\mathbb{R}^1)$ (use the L^2-theory near the support of a and use the cancellation property near infinity). It is not completely

clear, however, from this atomic definition that Ca is, in fact, in $H^1(\mathbb{R}^1)$ (although it is true). Ca is not an atom since it does not have compact support. Nevertheless, it does have mean zero and has rapid decay at infinity. One can show that Ca is a sum of atoms which have as their supports the support B of a and the successive doubles of B. Furthermore, the coefficients in this sum of atoms decrease as fast as the terms of a geometric series. This configuration, an element of $H^1(\mathbb{R})$ which can be realized as a sum of "neatly stacked" atoms, which we call a molecule, occurs frequently. In fact, many of the operators to which we have alluded, that are bounded on H^1 but not on L^1, have the property that they map atoms uniformly into molecules. This fact (and its generalizations, for example to H^p, $p < 1$) is one of the major themes of the second paper in this volume. The fundamental size and cancellation conditions which characterize molecules are captured by the following:

Definition. A function $M \in L^2(\mathbb{R}^n)$ is a molecule centered at 0 (for $H^1(\mathbb{R}^n)$) if

$$\left(\int_{\mathbb{R}^n} |M(x)|^2 dx\right)\left(\int_{\mathbb{R}^n} |M(x)|^2 |x|^{n+1} dx\right)^n \leq C \quad \text{and} \quad \int_{\mathbb{R}^n} M(x)dx = 0 .$$

It is a direct verification that an atom is a molecule. Conversely, as indicated above, every molecule can be written as a neatly stacked sum of atoms.

The definition just given is well suited to analysis by Fourier transform techniques. For instance, if $n = 1$, then $\hat{M}$, the Fourier transform of the molecule M, is characterized by

$$\left(\int_{\mathbb{R}} |\hat{M}(\xi)|^2 d\xi\right)\left(\int_{\mathbb{R}} |\hat{M}'(\xi)|^2 d\xi\right) \leq C , \quad \hat{M}(0) = 0 .$$

It is straightforward to check that if a is an atom and m is a function satisfying $\|m\|_\infty + \|\xi m'(\xi)\|_\infty < \infty$ then $\hat{M} = m\hat{a}$ satisfies the above condition. These facts form the outline of the proof that these multiplier operators map H^1 boundedly into itself.

Let us now pass to the other situation where the building blocks consist of