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CORRADO DE CONCINI
DAVID EISENBUD
CLAUDIO PROCESI

Hodge algebras

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Let k be a field and let A be a commutative k-algebra generated by some finite set $H = \{x_1,\ldots,x_n\} \subset A$. As a vectorspace, A will of course admit as basis a set of monomials in the $x_1,\ldots,x_n$, though this set is far from unique. It is even possible to arrange H in such a way that this basis set can be taken to be the complement of an ideal Σ of monomials (see section 1 for a precise definition), so that A has a basis in 1-1 correspondence with the natural monomial basis of a ring of the form

$$A_0 = k[x_1,\ldots,x_n]/I,$$

where I is generated by the monomials Σ in the variables $x_1,\ldots,x_n$, but in general the relation between A and A_0 will be very slight.

In this paper we consider an additional condition on Σ, H, and A, which slightly limits the multiplication in A, in terms of a partial order on H (Section 1); if the condition is satisfied, we say that A is a <u>Hodge Algebra</u>, governed by Σ. If this condition is satisfied, then (among other things) the relation between A and A_0 becomes very precise: A is, in a very special way, the "general fiber" of a flat deformation whose special fiber is A_0, so that many properties of A_0 may be transferred to A (Section 3). Thus when A is a Hodge algebra governed by some "good" Σ, many properties of A can be read off directly.

Many interesting examples turn out to be Hodge algebras governed by "good" ideals Σ, so results of the above type may be used to unify and extend a large amount of information about difficult concrete examples, such as coordinate rings of Grassmannians and certain generalized Grassmannians, flag manifolds, Schubert varieties, determinantal and Pfaffian varieties, varieties of minimal degree, and varieties of complexes.

History

The first explicit description of an interesting algebra via a basis of monomials and relations of our type that we know of is Hodge's study of the Grassmann variety and Schubert cycles [Hodge], undertaken with a view to obtaining explicit "postulation formulas" (in modern terms, the Hilbert functions of the homogeneous coordinate rings); the results are presented in a relatively readable way at the end ot volume 2 of [Hodge-Pedoe]. It is because of this, following a suggestion of Laksov, that we have called the algebras here "Hodge Algebras". Igusa, in [Igusa (1)] also exploited what is in fact a Hodge-algebra structure in proving the projective normality of the Grassmann variety. The next occurrence we know of is the "straightening law" of [Doubilet - Rota-Stein]; this was re-proved in [De Concini-Procesi] and [De Concini-Eisenbud-Procesi], where it is shown that this "straightening law" may be deduced in a simple way from the Hodge structure on the coordinate ring of the Grassmann variety. After [De Concini-Eisenbud-Procesi] was written, we made, at the suggestion of David Buchsbaum, a study of the relation between the Doubilet-Rota-Stein "straightening law" and the proof of the Cohen-Macaulayness of the Schubert cycles and determinantal varieties found in [Musili]; it was from this that the axioms for a Hodge algebra, in the special case called an "ordinal Hodge algebra" below, emerged (Musili's motivating proof, in our axiomatic form, is given in Section 8). This material was worked out by us in 1978, and a manuscript was then circulated; it is summarized in [Eisenbud].