Astérisque

B. A. KUPERSHMIDT

Discrete Lax equations and differential-difference calculus

Astérisque, tome 123 (1985)

<http://www.numdam.org/item?id=AST_1985__123__1_0>

123 ASTÉRISQUE

1985

DISCRETE LAX EQUATIONS AND DIFFERENTIAL-DIFFERENCE CALCULUS

by B. A. KUPERSHMIDT

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

A.M.S. Subjects Classification : 08 ; 34 A, K ; 39 ; 49 ; 70 G, H.

PREFACE

Since the discovery of solitons about 15 years ago, the classical theory of completely integrable systems has undergone remarkable transformation. Among many mathematical branches which benefited from this progress, the classical calculus of variations is one of the most conspicuous, being at the same time the most indispensable tool in the study of the structural problems.

For the continuous mechanical systems, the basic developments in both above mentioned theories are by now well known under the name of (differential) Lax equations (see, e.g., Manin's review [10]). Here I take up the case of classical mechanics proper but for the case of an infinite number of particles. It turns out that the appropriate calculus, resulting from an attempt to look at classical mechanics from the point of view of field theories, and not vice versa as is the custom, exists and in its logical structure, resembles very much the classical one though it does not have a geometric model.

The path of the presentation follows, as close as possible, the differential theory of Lax equations. A superficial familiarity with the latter will undoubtedly help the reader to understand the strings in various constructions, although I often supply the necessary motivation. There are no other prerequisites.

A few things have not found their way into the text. Most important among them are the matrix equations and their connections with simple Lie groups. This theory is at present largely unknown, however strange such a state of affairs may appear, especially in contrast with the presumably more complicated differential case, where the beautiful theory has been developed (see [12], [2], [14]). Want of space has led to the exclusion from the notes of the following topics which are of interest:

--Noncommutative calculus of variations which, in its differential part, stands in the same relation to the left invariant calculus of variations on a Lie group G as the Poisson structure on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of G stands to the left invariant part of the usual Hamiltonian formalism on the cotangent bundle T^*G.

--The restriction of a family of commuting flows to the stationary manifold of one of them, leading to the theory of the so-called "Finite Depth"-type equations.

--Generalized theorems on splitting and translated invariants for Lie algebras over function rings.

In order not to extend the size of the notes beyond the bounds of reason, I have omitted the most voluminous chapter X with proofs of the Hamiltonian property of a few quadratic and cubic matrices. The reader can reconstruct the proofs using methods of Chapter VIII (see also Chapter 1 in [10]).

These notes are an expanded version of lectures delivered at the Centre de Mathématique de l'École Normale Supérieure in the spring of 1982. I am very grateful to J.-L. Verdier for the invitation to lecture and I am much indebted to him for very stimulating discussions of the problem of deformations. My thanks go to friends and colleagues who read various parts of the manuscript: J. Gibbon, J. Gibbons, A. Greenspoon, D. Holm, S. Omohundro, and especially M. Hazewinkel who suggested numerous improvements.

The material on τ-function in the last section of Chapter IX owes much to the talks with H. Flaschka in August 1982 during my visit to Tucson. The rest of the notes were written in the Spring-Summer of 1982 while I was at the Los Alamos National Laboratory. I am much indebted to the Center for Nonlinear Studies for its hospitality, and to M. Martinez for the speedy typing.