Astérisque

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Astérisque, tome 137 (1986)

<http://www.numdam.org/item?id=AST_1986__137__1_0>

137 ASTÉRISQUE

1986

PROJECTIVE GEOMETRY OF ELLIPTIC CURVES

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Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

A.M.S. Subjects Classification : 14K07, 14F05.

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Introduction

In this treatise we want to discuss some old and new topics concerning the projective geometry of elliptic curves embedded in some projective space $\mathbb{P}_n$. To be more precise, we want to study three different aspects of elliptic curves in $\mathbb{P}_n$, namely

1. The symmetries of elliptic normal curves
2. The Horrocks-Mumford vector bundle
3. The normal bundle of elliptic curves of degree 5.

These three subjects are closely related to each other and it is exactly this interrelation which we want to study. In order to give the reader some idea about what we intend to do, we want to outline the contents of the individual chapters.

In chapter I we shall study the symmetries of elliptic normal curves $C_n \subseteq \mathbb{P}_{n-1}$ of degree n. Translation by n-torsion points and involution of the curve C_n define $2n^2$ transformations of the projective curve C_n into itself, and they all lift to projective transformations of $\mathbb{P}_{n-1}$. We shall first define a suitable embedding (by means of specially chosen theta-functions which are products of translates of the Weierstrass σ-function), such that these symmetries take on a particularly simple form. This leads us to the Heisenberg group H_n in its Schrödinger representation. The material of this chapter is classically well known and the results can be traced back as far as to L. Bianchi [3] and A. Hurwitz [13].

If $n = p \geq 3$ is a prime number, then the symmetries of an elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$ lead to a special configuration of hyperplanes and projective subspaces of dimension $\frac{1}{2}(p-3)$. This configuration is of type $(p^2{}_{p+1}, p(p+1)_p)$ and generalizes the classical configuration of the points of inflection of a plane cubic (the case $p = 3$). We shall study this configuration in chapter II. Although it can already be found in a paper by C. Segre [16], it was only fairly recently that I discovered this. I first heard about this configuration from W. Barth. In any case, our construction is quite different from C. Segre's. We shall briefly come back to C. Segre's point of view in chapter IX.

In chapter III we shall discuss elliptic normal curves of degree 3,4 and 5 in order to illustrate the results of the previous chapters.

Chapter IV deals with the quadric hypersurfaces which go through a fixed elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$. We shall first give a simple proof of a special case of a theorem of Mumford [15] on abelian varieties. We shall show that every elliptic curve $C_n \subseteq \mathbb{P}_{n-1}$ of degree $n \geq 4$ is the scheme-theoretic intersection of quadrics of rank 3. Then we shall use the symmetries of elliptic normal quintics $C_5 \subseteq \mathbb{P}_4$ to find quadratic equations for these curves. The rest of this chapter deals with the singular quadrics through a given elliptic normal quintic. The main result is, that there exists a 1-dimensional family of rank 3 quadrics through C_5 whose singular lines form a ruled surface F of degree 15. The surface F is birational to the second symmetric product S^2C of C_5 and we shall construct an explicit map between S^2C and F. The methods used here go back to Ellingsrud and Laksov [6]. Finally we shall briefly explain the relation between the curves C_5 and Shioda's modular surface S(5).

The normal bundle N_C of an elliptic normal quintic $C_5 \subseteq \mathbb{P}_4$ is the main object of chapter V. We shall first prove that N_C is indecomposable. It is then an easy consequence of Atiyah's classification [1] of vector bundles over an elliptic curve to describe the normal bundle N_C explicitly. We shall use this to give another proof of a vanishing result originally due to Ellingsrud and Laksov [6]. This vanishing result will be essential for chapter VIII.

In chapter VI we shall return to the Heisenberg group H_n and study its natural operation on the space of homogeneous forms of degree n in n variables. For every prime number $p = n \geq 3$ we shall determine the dimension of the space of invariant forms. In particular, if $p = 5$, we find that

$$\dim \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)) = 6$$

a result which was first proved by Horrocks and Mumford in [9] where it played an essential role in the study of the Horrocks-Mumford bundle. One can easily give a basis of the space of invariant quintic forms in terms of the configuration studied in chapter II.

For application in chapter VIII we shall finally construct a basis of the 3-dimensional space of invariant quintic forms whose corresponding hypersurfaces are singular along C_5.

In chapter VII we shall explain the relation between the Horrocks-Mumford bundle F on $\mathbb{P}_4$ and elliptic normal quintics. We shall prove, that, if $C_5 \subseteq \mathbb{P}_4$ is an elliptic normal quintic embedded as described in chapter I, then there exists a unique section $s \in \Gamma(F)$ whose zero-set is (scheme-theoretically) the tangent surface Tan C_5. In other words, the Horrocks-Mumford bundle can be reconstructed from the tangent developable of C_5 by means of the Serre-construction. This makes the statement of [9, p. 79(a)] precise and supplies a proof at the same time.

The main objective of chapter VIII is the study of the normal bundle of elliptic space curves of degree 5. Every such curve is the projection of an elliptic normal curve $C_5 \subseteq \mathbb{P}_4$. The normal bundle of these curves was first classified by Ellingsrud and Laksov in their paper [6] which was the starting point for this work. The main point is that their classification uses a certain 1-parameter family of quintic hypersurfaces $Y_M \subseteq \mathbb{P}_4$. (For a precise statement see (VIII. 2.7)). We shall first recall the results of Ellingsrud and Laksov and then turn to the hypersurfaces Y_M. To describe and understand this family was my original motive for this work. We shall see that the Y_M form a linear family of quintic hypersurfaces, whose base locus consists of the union of the tangent surface Tan C_5 and the ruled surface F which we have studied in chapter IV. This enables us to characterize the 2-dimensional space $U \subseteq \Gamma(\mathcal{O}_{\mathbb{P}_4}(5))$ which belongs to the linear family Y_M. We shall first of all see that the elements of U are invariant under the Heisenberg group H_5. Moreover, U consists exactly of those H_5-invariant quintic forms which vanish on the tangent surface Tan C_5 and whose associated hypersurfaces are singular along C_5, i.e.

$$U = \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5)) \cap \Gamma_H(\mathcal{J}_C^2(5)).$$

We shall then relate this description to the Horrocks-Mumford vector bundle. Finally we shall describe U explicitly as a subspace of