ROLAND QUÊME

On diophantine approximation by algebraic numbers of a given number field : a new generalization of Dirichlet approximation theorem

Astérisque, tome 198-199-200 (1991), p. 273-283

<http://www.numdam.org/item?id=AST_1991__198-199-200__273_0>

ON DIOPHANTINE APPROXIMATION BY ALGEBRAIC NUMBERS OF A GIVEN NUMBER FIELD : A NEW GENERALIZATION OF DIRICHLET APPROXIMATION THEOREM

by

Roland QUÊME

Introduction

It is well known that for all $\alpha \in \mathbb{R}$, $\alpha \notin \mathbb{Q}$ there are infinitely many p/q, $|p|$, $q \in \mathbb{N}$ such that $|\alpha - p/q| < 1/q^2$ (Dirichlet theorem), and that for any real algebraic number $\alpha \notin \mathbb{Q}$ and for any $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exist only finitely many p/q, $|p|$, $q \in \mathbb{N}$ such that $|\alpha - p/q| < 1/q^{2+\varepsilon}$ (Roth theorem).

Let K be a number field of degree n, signature (r, s) and absolute value of discriminant D.
Let B be the Minkowski constant of K ($B = (4/\pi)^s \,.\, (n!/n^n) \,.\, \sqrt{D}$).
Let $\sigma : K \to \mathbb{R}^r \times \mathbb{C}^s$ be the embedding defined by :

$$\sigma(\rho) = (\sigma_1(\rho), \ldots, \sigma_r(\rho), \sigma_{r+1}(\rho), \ldots, \sigma_{r+s}(\rho))$$

where, as usually, $K = \sigma_1(K)$.

For $x, y \in \mathbb{R}^r \times \mathbb{C}^s$ we note $x = (x_j,\ j = 1, \ldots, r+s)$. Then we note $x + y = (x_j + y_j,\ j = 1, \ldots, r+s)$ and $x.y = (x_j.y_j,\ j = 1, \ldots, r+s)$.
We define, for $x \in \mathbb{R}^r \times \mathbb{C}^s$, the distance function and the norm function :

$$d(x) = |x_1| + \cdots + |x_r| + 2|x_{r+1}| + \cdots + 2|x_{r+s}|,$$
$$N(x) = |x_1| \cdots |x_r| . |x_{r+1}|^2 \cdots |x_{r+s}|^2 \,.$$

Let A be the ring of integers of K.

Then we obtain the diophantine approximation theorems :

(i) For $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$, there exist infinitely many $\beta = p/q$, $p, q \in A$ such that $0 < d(\alpha\sigma(q) - \sigma(p)) < n^2 . B^{2/n}/d(\sigma(q))$, with arbitrary large distance $d(\sigma(q))$.

(ii) For $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$, $\alpha_j \notin \sigma_j(K)$, $j = 1, 2, \ldots, r+s$, there exist infinitely many $\beta = p/q$, $p, q \in A$ such that $0 < N(\alpha - \sigma(p/q)) < (B/N_{K/\mathbb{Q}}(q))^2$.

We first summarize the state of the art with three types of generalizations found in the quoted literature for diophantine approximation by numbers of a given number field K. Let K be a number field of degree n, signature (r, s). For $\beta \in K$, let $P(\beta)$ be the field polynomial of β,

$$P(\beta) = (x - \sigma_1(\beta)) \cdots$$
$$(x - \sigma_r(\beta))(x - \sigma_{r+1}(\beta))\overline{(x - \sigma_{r+1}(\beta))} \cdots (x - \sigma_{r+s}(\beta))\overline{(x - \sigma_{r+s}(\beta))} \ .$$

Let $C \in \mathbb{N}$ such that $P_1(\beta) = CP(\beta) = b_n\beta^n + \cdots + b_1\beta + b_0$ is a polynomial with integer coprime coefficients b_i, $i = 0, 1, \ldots, n$. Then we define the height of $\beta \in K$ by $H_K(\beta) = \sup_{i=0,\ldots,n} |b_i|$.

The first generalization of Dirichlet theorem found in bibliography is :

Assume that $r > 0$ and choose a real embedding $\sigma_1 : K \to \mathbb{R}$. For every $\alpha \in \mathbb{R} - \sigma_1(K)$, then there exist infinitely many $\beta \in K$ such that $|\alpha - \sigma_1(\beta)| < C_1(K)\max(1, \alpha^2)/H_K(\beta)^2$ where $C_1(K)$ is a constant depending only on K (see SCHMIDT [8] p.253).

The second generalization of Dirichlet theorem is :

Assume that $s > 0$ and choose a complex embedding $\sigma_2 : K \to \mathbb{C}$. For every $\alpha \in \mathbb{C} - \sigma_2(K)$, then there exist infinitely many $\beta \in K$ such that $|\alpha - \sigma_2(\beta)| < C_2(K)/H_K(\beta)$ where $C_2(K)$ is a constant depending only on K (see SCHMIDT [6] p.206).

The third generalization is :

Let $\beta_1, \ldots, \beta_\ell \in K$; let $\mathfrak{b}$ be the fractional ideal of K generated by $(1, \beta_1, \ldots, \beta_\ell)$.

We define the generalized height of the ℓ-tuple $(\beta_1, \ldots, \beta_\ell)$ by :

$$\mathfrak{h}_K(\beta_1, \ldots, \beta_\ell) = N_{K/\mathbb{Q}}(\mathfrak{b}) \prod_{j=1}^{r} \max(1, |\sigma_j(\beta_1)|, \ldots, |\sigma_j(\beta_\ell)|) \prod_{j=r+1}^{r+s} \max(1, |\sigma_j(\beta_1)|, \ldots, |\sigma_j(\beta_\ell)|)^2.$$

(i) if $r > 0$, let $\sigma_3 : K \to \mathbb{R}$ be a real embedding and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$, not all in $\sigma_3(K)$; put in that case $\nu = 1$;

(ii) if $s > 0$, let $\sigma_3 : K \to \mathbb{C}$ be a complex embedding and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{C}$, not all in $\sigma_3(K)$; put in that case $\nu = 2$;

then there is a constant $C_3(K, \alpha_1, \ldots, \alpha_\ell)$ depending only on $K, \alpha_1, \ldots, \alpha_\ell$ such that there exist infinitely many $\beta = (\beta_1, \ldots, \beta_\ell)$, $\beta_i \in K$, with

$$|\alpha_i - \sigma_3(\beta_i)|^\nu < C_3(K, \alpha_1, \ldots, \alpha_\ell).\mathfrak{h}_K(\beta_1, \ldots, \beta_\ell)^{-1-1/\ell}, \ i = 1, 2, \ldots, \ell \quad (1)$$

(see SCHMIDT [7] p.2).

The main difference between the quoted formulation and our theorem are summarized in the four next points :

1) In classical approximations above, $|\alpha - \beta|$ is obtained for *one* of the conjugates $\beta = \sigma_1(\beta)$. On the other hand, our estimate involves simultaneously *all* the conjugates of the same $\beta \in K$,

   for the distance function,

$$d(\alpha\sigma(q) - \sigma(p)) = |\alpha_1\sigma_1(q) - \sigma_1(p)| + \cdots + |\alpha_r\sigma_r(q) - \sigma_r(p)| + 2|\alpha_{r+1}\sigma_{r+1}(q) - \sigma_{r+1}(p)| + \cdots + 2|\alpha_{r+s}\sigma_{r+s}(q) - \sigma_{r+s}(p)|$$

   for the norm function,

$$N(\alpha - \sigma(p/q)) = |\alpha_1 - \sigma_1(p/q))| \cdots |\alpha_r - \sigma_r(p/q)|.|\alpha_{r+1} - \sigma_{r+1}(p/q)|^2 \cdots |\alpha_{r+s} - \sigma_{r+s}(p/q)|^2.$$

2) Our approximation theorem cannot be immediately connected to usual simultaneous approximation theorems, because in simultaneous approximation $|f(\alpha_1 - \beta_1)|, \ldots, |f(\alpha_\ell - \beta_\ell)|$ the simultaneous approximations $\beta_1, \ldots, \beta_\ell$ are not conjugate of the same $\beta \in K$ (see for instance (1)).

3) Our result contains not only effective but *explicit* constants with *simple* relationship to the structure of the number fields (the Minkowski constant for instance, with the distance function choosen).

4) Our proof is the exact generalization of the approximation by $\mathbb{Q}$ to approximation by a given number field K, using geometry of numbers properties of number fields embedding in $\mathbb{R}^n$.

Acknowledgments are due to Professors DUBOIS, GYÖRY, LEUTBECHER, SCHLICKEWEI and TOFFIN for helpful remarks which allowed me to write this new version of this note.

Prerequisites-Notations

K : number field

n : degree of K

(r, s) : signature of K

x : $x \in \mathbb{R}^r \times \mathbb{C}^s$, $x = (x_j \mid j = 1, \ldots, r+s)$

$x + y$: $x + y = (x_j + y_j \mid j = 1, \ldots, r+s)$

$x.y$: $x.y = (x_j.y_j \mid j = 1, \ldots, r+s)$

$d(x)$: for $x \in \mathbb{R}^r \times \mathbb{C}^s$, the distance function is defined by :

$$d(x) = |x_1| + \cdots + |x_r| + 2|x_{r+1}| + \cdots + 2|x_{r+s}|$$

$N(x)$: for $x \in \mathbb{R}^r \times \mathbb{C}^s$, the norm form is defined by :

$$N(x) = |x_1| \cdots |x_r|.|x_{r+1}|^2 \cdots |x_{r+s}|^2$$

$U(o, \tau)$: for $\tau \in \mathbb{R}_+$, convex body of $\mathbb{R}^n$ defined by

$$U(o, \tau) = \{x \mid x \in \mathbb{R}^r \times \mathbb{C}^s, d(x) < n\tau\}$$

where $\mathbb{R}^r \times \mathbb{C}^s$ is isomorphically identified to $\mathbb{R}^n$ by

$$x_{r+i} = (R(x_{r+i}), I(x_{r+i})) \ , \ i = 1, \ldots, s \ ,$$