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PROPAGATION OF SINGULARITIES IN THREE-BODY SCATTERING

András Vasy

Abstract. — In this paper we consider a compact manifold with boundary X equipped with a scattering metric g and with a collection C_i of disjoint closed embedded submanifolds of ∂X. Thus, g is a Riemannian metric in $\text{int}(X)$ of the form $g = x^{-4}\,dx^2 + x^{-2}h$ near ∂X for some choice of a boundary defining function x, h being a smooth symmetric 2-cotensor on X which is non-degenerate when restricted to ∂X. We also let Δ be the (positive) Laplacian of g, suppose that $V \in \mathcal{C}^\infty([X; \cup_i C_i])$ where $[X; \cup_i C_i]$ is X blown up along the C_i, assume that V vanishes at the lift of ∂X, and consider the operator $H = \Delta + V$. Three-body scattering with smooth potentials which have an asymptotic expansion at infinity (possibly Coulomb-type) provide the standard example of this setup. We analyze the propagation of singularities of generalized eigenfunctions of H, showing that this is essentially a hyperbolic problem which has much in common with the Dirichlet and transmission problems for the wave operator, though additional features arise due to the presence of bound states of the 'two-body operators'. We also show that the wave front relation of the free-to-free part of the scattering matrix is given by the broken geodesic flow at distance π.

Résumé **(Propagation des singularités dans la diffusion à trois corps)**

Dans cet article, nous considérons une variété X compacte à bord, munie d'une métrique de diffusion g et d'une famille C_i de sous-variétés fermées de ∂X deux à deux disjointes. Ainsi, g est une métrique riemannienne dans $\text{int}(X)$ de la forme $g = x^{-4}\,dx^2 + x^{-2}h$ près de ∂X, pour un choix convenable de fonction x définissant le bord, h étant un 2-cotenseur symétrique $\mathcal{C}^\infty$ sur X qui est non dégénéré en restriction à ∂X. Nous notons aussi Δ le laplacien (positif) de g, choisissons $V \in \mathcal{C}^\infty([X; \cup_i C_i])$, où $[X; \cup_i C_i]$ est X éclaté le long des C_i, tel que V s'annule sur le relevé de ∂X, et nous considérons l'opérateur $H = \Delta + V$. Les diffusions à trois corps avec potentiels $\mathcal{C}^\infty$ qui ont un développement asymptotique à l'infini (éventuellement du type de Coulomb) sont des exemples de cette situation. Nous analysons la propagation des singularités des fonctions propres généralisées de H en montrant que c'est essentiellement un problème hyperbolique qui a beaucoup en commun avec les problèmes de Dirichlet et de transmission pour l'opérateur des ondes, avec néanmoins des propriétés supplémentaires dues à la présence d'états bornés des opérateurs à deux corps. Nous

montrons aussi que la relation de front d'onde de la partie libre-libre de la matrice de diffusion est donnée par le flot des géodésiques brisées à distance π.

CONTENTS

CHAPTER 1

INTRODUCTION

Let X be a compact manifold with boundary. In **[25]** Melrose has defined the algebra $\mathrm{Diff}_{\mathrm{sc}}(X)$ of scattering differential operators on X. In fact, let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function of X, so $x \geq 0$, $dx \neq 0$ on ∂X, and $\partial X = \{x = 0\}$. The Lie algebra of b-vector fields on X, $\mathcal{V}_{\mathrm{b}}(X)$, is the set of all smooth vector fields on X which are tangent to ∂X. The Lie algebra of scattering vector fields on X, $\mathcal{V}_{\mathrm{sc}}(X)$, is simply $\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X)$; this notion is independent of the choice of the boundary defining function x. Much as in the case of $\mathcal{V}_{\mathrm{b}}(X)$, $\mathcal{V}_{\mathrm{sc}}(X)$ is the set of all smooth sections of a vector bundle over X; this bundle is denoted by ${}^{\mathrm{sc}}TX$. Finally, $\mathrm{Diff}_{\mathrm{sc}}(X)$ is just the enveloping algebra of $\mathcal{V}_{\mathrm{sc}}(X)$, i.e. the ring of operators on $\mathcal{C}^\infty(X)$ generated by $\mathcal{C}^\infty(X)$ (considered as multiplication operators) and $\mathcal{V}_{\mathrm{sc}}(X)$. An example of such an operator is the Laplacian Δ associated to a scattering metric g. Thus, g is a Riemannian metric in $\mathrm{int}(X)$ of the form $g = x^{-4}\,dx^2 + x^{-2}h$ near ∂X for some choice of a boundary defining function x, h being a smooth symmetric 2-cotensor on X which is non-degenerate when restricted to ∂X. In particular, g is a metric on ${}^{\mathrm{sc}}TX$.

Let C_i, $i = 1, \dots, k$, be disjoint closed embedded submanifolds of ∂X. Here the C_i might have different dimensions. Nevertheless, to simplify the notation, we introduce $C = \cup_i C_i$, and say that C is also a closed embedded submanifold of ∂X, although this is strictly speaking only true if the dimensions of the connected components of C are the same. Let mf ('main face') be the lift of ∂X to $[X; C]$, the blow-up of X along C (see the Appendix of **[25]** for a treatment of blow-ups, and see Figure 1 for a picture). We write ρ_{mf} for a defining function of mf. The 'three-body type' operators we are interested in are perturbations H of Δ of the form $H = \Delta + V$, where $V \in \mathcal{C}^\infty([X; C])$ is real-valued and vanishes at mf. As discussed in the following paragraphs, three-body Hamiltonians, with the center of mass removed, give an example of such operators, and explain our interest in the problem. In the degenerate case when $k = 0$, i.e. $C = \varnothing$, we arrive at the generalized 'two-body type' scattering considered in Melrose's original paper **[25]**; in this case $V \in x\mathcal{C}^\infty(X)$.

Consider the Euclidian space, $\mathbb{R}^N$, with the standard metric, and its radial compactification to the upper hemisphere $\mathbb{S}^N_+$. Embedding $\mathbb{S}^N_+$ in $\mathbb{R}^{N+1}$ as the unit upper hemisphere this is given by the map $\mathrm{SP} : \mathbb{R}^N \to \mathbb{S}^N_+$

$$\mathrm{SP}(z) = \left(\frac{1}{(1+|z|^2)^{1/2}}, \frac{z}{(1+|z|^2)^{1/2}}\right). \tag{1.1}$$

Let x be a boundary defining function of $\mathbb{S}^N_+$ such that $x = (\mathrm{SP}^{-1})^*|z|^{-1}$ near $\partial\mathbb{S}^N_+$. Then the Euclidian metric pulls back to a scattering metric on $\mathbb{S}^N_+$, with h being the standard metric on $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$, and the Euclidian Laplacian becomes an element of $\mathrm{Diff}^2_{\mathrm{sc}}(\mathbb{S}^N_+)$.

Let X_i, $i = 1, \ldots, k$, be linear subspaces of $\mathbb{R}^N$, let X^i be the orthocomplement of X_i, $n_i = \dim X^i$, and let π^i be the orthogonal projection to X^i. By a Euclidian many-body Hamiltonian we mean an operator of the form $H = \Delta + \sum_i (\pi^i)^* V_i$ where $V_i \in \mathcal{C}^\infty(X^i; \mathbb{R})$ satisfy $(\mathrm{SP}_i^{-1})^* V_i \in \rho_i \mathcal{C}^\infty(\mathbb{S}^{n_i}_+)$ with ρ_i denoting a boundary defining function of $\mathbb{S}^{n_i}_+$, and SP_i being the radial compactification map $\mathrm{SP}_i : X^i \to \mathbb{S}^{n_i}_+$. The condition on V_i means that it is a one-step polyhomogeneous symbol on X^i of order -1. A Euclidian three-body Hamiltonian (with center of mass removed) is a many-body Hamiltonian with the additional assumption that $X_i \cap X_j = \{0\}$ for $i \neq j$. In the compactified picture, writing $\overline{X_i} = \mathrm{cl}(\mathrm{SP}(X_i)) \subset \mathbb{S}^N_+$, $C_i = \overline{X_i} \cap \mathbb{S}^{N-1}$, the condition $X_i \cap X_j = \{0\}$ for $i \neq j$ becomes $C_i \cap C_j = \varnothing$ for $i \neq j$. With the notation $C = \cup_i C_i$ as in the general case, it is straightforward to check that

$$V = (\mathrm{SP}^{-1})^* \sum_i (\pi^i)^* V_i \in \mathcal{C}^\infty([\mathbb{S}^N_+; C]), \qquad V|_{\mathrm{mf}} = 0 \tag{1.2}$$

(this will be done here in Lemma 2.1), so H is indeed a 'three-body type' operator as described above in the geometric setting. Note also that the C_i are 'subspheres' of $\mathbb{S}^{N-1}$, in particular, they are totally geodesic with respect to the standard metric. A two-body Hamiltonian corresponds to taking $k = 1$, $X_1 = \{0\}$ above, so we have $V \in x\mathcal{C}^\infty(\mathbb{S}^N_+)$, giving rise to the 'two-body type' terminology in the geometric setting. In Figure 1 below we take $N = 2$ and the X_i are lines. Hence, $X = \mathbb{S}^2_+$ is a disk, $\partial X = \mathbb{S}^1$, each C_i consists of two points. The lift of C_i to $[X; C]$ is denoted by ff_i in the figure.

Now we return to the general setting. First note that $H = \Delta + V$ is self-adjoint on $L^2_{\mathrm{sc}}(X)$, the L^2 space defined by integration with respect to the Riemannian density dg, since Δ and V are such and V is bounded. Hence, its resolvent $R(\lambda) = (H - \lambda)^{-1}$ is a bounded linear operator on $L^2_{\mathrm{sc}}(X)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In this paper we analyze the boundary value of the resolvent at the real axis, i.e. $R(\lambda \pm i0)$. We show that $\mathrm{spec}_p(H) \cap (0, \infty) = \varnothing$ and

$$R(\lambda \pm i0) \in \mathcal{B}(x^{1/2+\epsilon} L^2_{\mathrm{sc}}(X), x^{-1/2-\epsilon} L^2_{\mathrm{sc}}(X)) \tag{1.3}$$

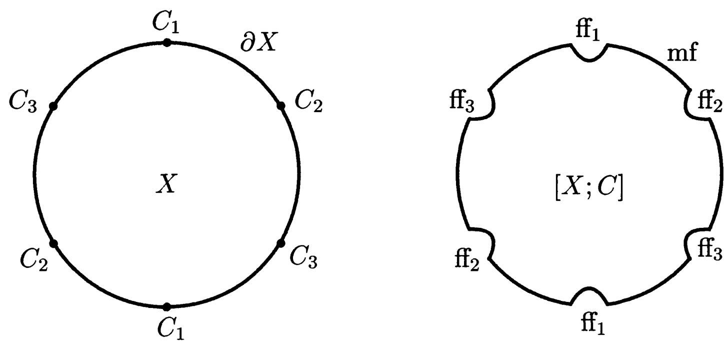

FIGURE 1. The original space X and its resolution $[X; C]$.

for all $\varepsilon > 0$. This is completely analogous to the classical result of Mourre in Euclidian three-body scattering (**[30, 31]**, see also the paper **[32]** of Perry, Sigal and Simon in which they extend Mourre's results to many-body systems), together with the absence of positive eigenvalues which was shown by Froese and Herbst **[8]** in the Euclidian case.

We also show that for $f \in \dot{\mathcal{C}}^\infty(X)$, $R(\lambda \pm i0)f$ has a complete asymptotic expansion away from C which is similar to the corresponding expansion for Euclidian two-body Hamiltonians. Here $\dot{\mathcal{C}}^\infty(X)$ is the subspace of $\mathcal{C}^\infty(X)$ consisting of functions which vanish at ∂X together with all of their derivatives. For simplicity here we only state the asymptotic expansion if $V \in \rho_{\mathrm{mf}}^2 \mathcal{C}^\infty([X; C])$ (i.e. short-range); the general case is described in Theorem 18.6. It is convenient to replace the spectral parameter λ by λ^2. Then, for $\lambda > 0$, $f \in \dot{\mathcal{C}}^\infty(X)$, the expansion can be described by

$$v_\pm = e^{\pm i\lambda/x} x^{-(N-1)/2} R(\lambda^2 \mp i0) f \in \mathcal{C}^\infty(X \setminus C). \tag{1.4}$$

The top term of such an expansion for Euclidian three-body scattering was described by Isozaki in **[21]**, assuming that the potentials were short range, by Herbst and Skibsted in **[16]** in the long-range many-body Euclidian case, and the full expansion was proved by the author in **[38]**. Moreover, we show that given any 'initial data' $a_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$ we can find $f \in \dot{\mathcal{C}}^\infty(X)$ such that with v_- as above we have $v_- \in \mathcal{C}^\infty(X)$ and $a_0 = v_-|_{\partial X}$. Then

$$u = R(\lambda^2 + i0) f - R(\lambda^2 - i0) f \in \mathcal{C}^{-\infty}(X) \tag{1.5}$$

satisfies $(H - \lambda)u = 0$, and has the form

$$u = e^{i\lambda/x} x^{(N-1)/2} v_- - e^{-i\lambda/x} x^{(N-1)/2} v_+. \tag{1.6}$$

For $\lambda > 0$ the Poisson operator corresponding to 'free initial data' is the map $P(\lambda) : \mathcal{C}_c^\infty(\partial X \setminus C) \to \mathcal{C}^{-\infty}(X)$ given by $P(\lambda)a_0 = u$. This definition is justified by the uniqueness statement of Theorem 19.1 which is again an analog of Isozaki's result **[22]**. The free-to-free part of the scattering matrix, $S(\lambda)$, relates the leading part of