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Abstract. — In this paper, we formulate and prove a duality theorem for the equivariant stable homotopy category, using the language of Verdier duality from sheaf theory. We work with the category of G-equivariant spectra (for a compact Lie group G) parametrized over a G-space X, and consider a smooth equivariant family $f : X \to Y$, which is a G-equivariant bundle whose fiber is a smooth compact manifold, and with actions of subgroups of G varying smoothly over Y. Then our main theorem is a natural equivalence between a certain direct image functor f_* and a "direct image with proper support functor" $f_!$, in the stable equivariant homotopy category over Y. In particular, the Wirthmüller and Adams isomorphisms in equivariant stable homotopy theory turn out to be special cases of this duality theorem.

Résumé **(Dualité pour les familles lisses en théorie de l'homotopie stable équivariante)**
Dans cet article, nous énonçons et démontrons un théorème de dualité pour la catégorie de l'homotopie stable équivariante, en utilisant le langage de la dualité de Verdier provenant de la théorie des faisceaux. Nous travaillons avec la catégorie des spectres G-équivariants (pour un groupe de Lie compact G) paramétrés par un G-espace X, et nous considérons une famille lisse équivariante $f : X \to Y$, c'est-à-dire un fibré G-équivariant de fibre une variété lisse compacte, et avec des actions de sous-groupes de G variant de manière lisse sur Y. Notre résultat principal est alors une équivalence naturelle entre un foncteur image directe f_* et un foncteur « image directe à support propre $f_!$ », dans la catégorie de l'homotopie stable équivariante sur Y. Les isomorphismes de Wirthmüller et Adams en théorie de l'homotopie stable équivariante apparaissent comme des cas particuliers de ce théorème de dualité.

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INTRODUCTION

The purpose of this paper is to formulate and prove a stable homotopy duality theorem for smooth equivariant families of manifolds, using a relationship of the stable homotopy language with sheaf theory. We work with G-equivariant spaces and spectra parametrized over G-equivariant spaces, where G is a compact Lie group. To relate this to the language of sheaves and Verdier duality from algebraic geometry (see *e.g.* [**2, 6**]), we introduce the notions of sheaves of spaces and of spectra. The Grothendieck site we use here is the most basic case, where the category is the comma category $G\mathrm{Top}\,/X$ of all G-equivariant topological spaces mapping to a given G-equivariant base space X. The coverings in this category are all colimits. This makes the results of this paper more directly related to classical stable homotopy theory [**8**] than its generalizations (*e.g.* [**1, 12**]), although our methods in principle also seem to apply to those more general contexts.

In our context, the main theorem is that for a map $f : X \to Y$ of base spaces satisfying certain conditions, there is a natural equivalence in the stable homotopy categories

$$f_* \simeq f_! \tag{0.1}$$

between a certain direct image functor f_* and a direct image with proper support functor $f_!$. This is an analogue of a classical result for proper maps of schemes, and abelian sheaves. A complementary statement for smooth maps relate the inverse image functor f^* to $f^!$, an inverse image with proper support functor in the derived category of abelian sheaves. We also have an analogue of this statement. As one would expect, our theorem implies Poincaré duality for equivariant manifolds (see [**8**]). It may perhaps be more surprising that it also includes other results of equivariant stable homotopy theory, namely the Wirthmüller and Adams isomorphisms [**8**].

We will work with maps f that are what we call equivariant smooth families of manifolds. Essentially, a G-equivariant map $f : X \to Y$ is an equivariant smooth family if it is an equivariant bundle whose fiber is a smooth compact manifold, and actions of subgroups of G on the fiber vary smoothly over the base space Y in a suitable sense (See Definition 4.2).

It turns out that in our case, instead of directly describing the direct image with proper support functor $f_!$, it is easier to define a left adjoint $f_\sharp$ to the inverse image functor f^*, and identify $f_!$ with $f_\sharp$ up to a shift by the *dualizing object* associated to the equivariant smooth family $f : X \to Y$. The dualizing object is a spectrum parametrized over X, which is invertible under the smash product in the homotopy category. A main part of the content of the theorem is to identify this dualizing object as the stable tangent bundle of X in the parametrized category over Y.

Another ingredient on which the meaning of our theorem depends heavily is the closed model structure on the categories of parametrized G-spaces and G-spectra. The duality theorem takes place in the homotopy category associated with the model structure on parametrized G-spectra. In Chapter 3, we give definitions of the model structures in detail. An important aspect of the model structure on parametrized spaces is that a G-space Z parametrized over X is fibrant if and only if the structure map $Z \to X$ is a fibration in the standard model structure on G-equivariant spaces (*i.e.* for fibrations, use Serre fibrations on H-fixed point sets for all closed subgroups H of G). A similar statement holds for parametrized G-spectra. Thus, one can think of the homotopy categories of parametrized G-spaces and spectra as dealing with objects that are in some sense bundle-like over the base space. (In particular, it does not capture objects such as skyscraper sheaves.)

We will show that the Wirthmüller and Adams isomorphisms are special instances of our duality theorem. Recall from [**8**] Theorem II.6.2 that for a (closed) subgroup H of G, the Wirthmüller isomorphism is that for an H-equivariant spectrum E

$$G \ltimes_H \Sigma^{-L} E \simeq F_H[G, E] \tag{0.2}$$

in the homotopy category of G-equivariant spectra. The two sides of the equivalence are the left and right adjoints to the forgetful functor from G-spectra to H-spectra, and the H-representation L is the tangent space of G/H at eH, with H-action by translation. If H is a normal subgroup of G, then the Adams isomorphism ([**8**] Theorem II.7.1) states that for an H-free G-spectrum E indexed on the H-fixed points $\mathcal{U}^H$ of a complete G-universe $\mathcal{U}$,

$$E/H \simeq (i_* E \wedge S^{-A})^H \tag{0.3}$$

in the homotopy category of G/H-equivariant spectra. Here, the two sides are the left and right adjoints to the functor from G/H-spectra to G-spectra that takes a G/H-spectrum to be an H-fixed G-spectrum. An H-free G-spectrum is a G-spectrum which has a cellular approximation, such that every cell is H-free, *i.e.* of the form $G/N_+ \wedge S^n$, where N is a subgroup of G such that $N \cap H = \{e\}$. The functor i_* from G-spectra indexed on $\mathcal{U}^H$ to G-spectra indexed on $\mathcal{U}$ is the universe change functor associated to the inclusion of universes $i : \mathcal{U}^H \to \mathcal{U}$ (see [**8**] Section II.1). Also, A is the adjoint representation of G, *i.e.* the tangent space of H at e, with G-action by conjugation.

The statement (0.2) of Wirthmüller isomorphism translates to the case of our duality theorem for the equivariant smooth family $f : G/H \to *$, via an equivalence of categories between H-equivariant spectra and G-equivariant spectra parametrized over G/H. The case of the Adams isomorphism is more complicated. The equivariant smooth family to which the duality theorem applies is the quotient map $f : E\mathcal{F} \to E\mathcal{F}/H$, where $E\mathcal{F}$ is the universal contractible H-free G-space, and $E\mathcal{F}/H$ its orbit space by H ([**8**] Section II.2). The closed model structures give an equivalence of homotopy categories between H-free G-spectra and G-spectra parametrized over $E\mathcal{F}$. Via this equivalence and composition with certain other functors, the duality theorem gives (0.3).

The organization of the paper is as follows. In Chapter 1, we give a formulation of Verdier duality from the theory of sheaves, to give motivations for bringing in the language of sheaves. The next two chapters give the foundations on G-equivariant spaces and spectra over a base space that we need for the main theorem. Namely, in Chapter 2, we recall the definitions of G-equivariant spaces and spectra over a base space X, and show that they are equivalent to the categories of sheaves on $G\mathrm{Top}/X$. We also give certain basic constructions such as the smash product, and define the base change functors, which are associated with a map $f : X \to Y$ of base spaces. Chapter 3 gives a self-contained definition of the closed model structures on the categories of G-spaces and spectra parametrized over X.

In Chapter 4, we state the main theorem of the paper, given in terms of equivalences between base change functors in the stable homotopy categories, up to a shift by a certain dualizing object, for a class of "good" maps $f : X \to Y$. This class of maps is the class of smooth families, which are G-equivariant bundles whose fibers are smooth manifolds. We also define the dualizing object, and prove some preliminary results towards proving the main theorem. The main part of the proof of the theorem is given in Chapter 5. For a smooth family $f : X \to Y$ where Y is compact, we define natural transformations between the base change functors on the level of spaces, which turn out to be homotopy inverses. Stabilizing gives the theorem in the case of a compact Y, and the general case is obtained via a colimit argument. In Chapter 6, we show that both the Wirthmüller and the Adams isomorphisms are examples of the main duality theorem. Finally, in Chapter 7, we give the proofs of some technical results on the closed model structure for G-spectra parametrized over X.