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ON BASIC PIECES OF AXIOM A DIFFEOMORPHISMS ISOTOPIC TO PSEUDOANOSOV MAPS

by

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Abstract. — We consider Axiom A diffeomorphisms g in the isotopy class of a pseudoanosov map f. It is shown that they have a unique "large" basic piece Λ, and necessary and sufficient conditions for g to be semiconjugated to f, that only involve conditions on Λ, are obtained. As a consequence, it is proved that if Λ is exteriorly situated, stable and unstable half-leaves of points of Λ boundedly deviate from geodesics.

1. Introduction

In this paper we consider Axiom A diffeomorphisms in the isotopy class of pseudoanosov maps. It is known (see [**H1, L1**]) that any pseudoanosov map f is persistent in its isotopy class i.e. for any homeomorphism g isotopic to f there exists a closed invariant set J_g such that g restricted to J_g is semiconjugated to f. When g verifies Axiom A, the first author (in [**L2**]) introduced the definition of "small" and "large" basic pieces of g. Roughly speaking, a basic piece B is small if an adequate lift to the universal cover of a B-stable (unstable) manifold is bounded. Otherwise, B is large. In [**L2**] it is proved that it is necessary and sufficient for a large basic piece Λ to be contained in J_g that the above mentioned lift lies at a bounded distance of an f-stable set of a point. In this work we prove that an Axiom A diffeomorphism g has a unique large basic piece Λ and, using that if g is not semiconjugated to f the Nielsen classes of g^n grow exponentially faster than the Nielsen classes of f^n (see [**H2**]), we prove that it is necessary and sufficient for g being semiconjugated to f that $\Lambda \subset J_g$. These results are proved in sections 3 and 4.

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In section 5 we lead with the case when the large basic set Λ is exteriorly situated i.e. there are no nul-homotopic loops that consist in the union of a stable and an unstable arc of a point of Λ. We show that, in this case, g is semiconjugated to f. As a consequence we obtain that half-leaves of stable and unstable manifolds of Λ boundedly deviates from geodesics that have the same asymptotic direction. Let us say that V. Grines (see [**G1**] and the survey [**G2**]) obtained this bounded deviation property for unstable (stable) manifolds of exteriorly situated nontrivial attractors (resp. repellers) of any diffeomorphism of a closed surface of genus larger than one and also for both, unstable and stable manifolds, provided the attractor (repeller) is a basic piece of a diffeomorphism that verifies Axiom A and that satisfies the strong transversality condition. In [**RW**] R. C. Robinson and R. Williams constructed an example of an Axiom A diffeomorphism on a surface of genus > 1 such that its nonwandering set consists exactly of an exteriorly situated attractor and an exteriorly situated repeller and that does not verify the strong transversality condition. V. Grines ([**G3**]) also showed that the stable manifold of the attractor (unstable for the repeller) of this example does not have the bounded deviation from geodesics property. Our results show that this kind of behaviour is impossible in the isotopy class of pseudoanosov maps (the fact that the Robinson-Williams example is not isotopic to a pseudoanosov map can be checked directly)

Finally, let us say that to understand the proofs, some familiarity with the theory of pseudoanosov maps is needed. We do not include this background material that the reader may find in [**FLP, CB, M, HT, T**].

2. Preliminaries

Let f be a pseudoanosov map of a compact connected oriented boundaryless surface M, let F be a lift of f to the universal cover $\mathcal{M}$ of M and $\pi : \mathcal{M} \to M$ the covering projection. Then, there exist (see [**T, FLP**]) equivariant pseudometrics D_S, D_U, and $\lambda > 1$ such that, for $\xi, \eta \in \mathcal{M}$,

$$D_S(F^{-1}(\xi), F^{-1}(\eta)) = \lambda D_S(\xi, \eta)$$
$$D_U(F(\xi), F(\eta)) = \lambda D_U(\xi, \eta)$$

and $D = D_S + D_U$ is an equivariant metric on $\mathcal{M}$.

For the remainder of this paper g will be a homeomorphism isotopic to f.

Definition 2.1. — *We say that $x, y \in M$ are equivalent iff for some (and then for any) lift G of g there exist $\xi \in \pi^{-1}(x)$, $\eta \in \pi^{-1}(y)$ and $K > 0$ such that $D(G^n(\xi), G^n(\eta)) \leqslant K$ for all $n \in \mathbb{Z}$.*

Obviously, this is an equivalence relation. In the following proposition we state some properties of this relation. For completeness we include the proofs that are essentially contained in [**H1, L1, CS**].

Proposition 2.2. — *The following statements hold:*

(1) *The constant K of the definition above only depends on g i.e. $\exists\, K_g$ such that if x is equivalent to y then $D(G^n(\xi), G^n(\eta)) < K_g$ for all $n \in \mathbb{Z}$. Moreover, K_g tends to 0 as g tends to f (see* [**L1, H1**]*).*

(2) *Let $[x]_g$ be the equivalence class of $x \in M$. Then, $[x]_g$ is a compact set and $g([x]_g) = [g(x)]_g$.*

(3) *The quotient space under the equivalence relation, $\overline{M}_g$, is a compact metrizable space and, by part 2, g induces a homeomorphism $\overline{g} : \overline{M}_g \to \overline{M}_g$. When $g = f$, $\overline{M}_f = M$ and $\overline{f} = f$, due to the infinite expansivity of any lift of f.*

Proof. — In order to prove the first part of the proposition take F, G to be adequate lifts of f, g and let $R > 0$ be such that $D(F, G) < R$ and $D(F^{-1}, G^{-1}) < R$.

Suppose that $D(\xi, \eta) > 2K$, then $D_U(\xi, \eta) > K$ or $D_S(\xi, \eta) > K$. With no loose of generality we can assume that the first inequality holds. If $K > 2R(\lambda - 1)^{-1}$ there exists $1 < \alpha < \lambda$ such that $K > 2R(\lambda - \alpha)^{-1} > 2R(\lambda - 1)^{-1}$ which implies $D_U(G(\xi), G(\eta)) > \lambda K - 2R > \alpha K$.

Thus, $D_U(G^n(\xi), G^n(\eta)) > \alpha^n K$. Since $\alpha^n K$ tends to infinite with n, this implies that if x and y are equivalent there exist ξ, η so that

$$D(G^n(\xi), G^n(\eta)) \leqslant 4R(\lambda - 1)^{-1} \quad \forall\, n \in \mathbb{Z}.$$

Choose $K_g = 4R(\lambda - 1)^{-1}$.

To prove the second part, take x_k equivalent to x for all $k \in \mathbb{N}$ and $x_k \to_{k \to +\infty} x^*$. Then, given $\xi \in \pi^{-1}(x)$, there are $\xi_k \in \pi^{-1}(x_k)$ such that $D(G^n(\xi_k), G^n(\xi)) < K_g$ $\forall\, n \in \mathbb{Z}$.

By taking, if necessary, a convergent subsequence, we may assume that $\xi_k \to_{k \to +\infty} \xi^* \in \pi^{-1}(x^*)$. Then, $D(G^n(\xi^*), G^n(\xi)) < K_g$ $\forall\, n \in \mathbb{Z}$.

Now we prove the third part of the proposition. The proof is similar to the one included in [**CS**] where the same result for g C^0-close enough to an expansive homeomorphisms is shown.

It is not difficult to see that if $\{x_n\} \subset M$ is a sequence such that $x_n \to x$, $\limsup [x_n]_g \subset [x]_g$. Then, given an open set $U \subset M$, the set $\{y \in M; [y]_g \subset U\}$ is open. This easily implies that $\overline{M}_g$ is Haussdorf and metrizable. □

We want to study the connection between the dynamics of $\overline{g}$ and f; to this end we find conditions on g to be semiconjugated to f.

Define

Definition 2.3. — *The g-orbit of x is shadowed by the f-orbit of y iff there exist $\xi \in \pi^{-1}(x), \eta \in \pi^{-1}(y)$ such that $\{D(G^n(\xi), F^n(\eta)) : n \in \mathbb{Z}\}$ is bounded, for G, F equivariantly homotopic lifts of g, f.*

Observe that if the g-orbit of x is shadowed by the f-orbit of y, then y is unique and every g-orbit of a point of $[x]_g$ is shadowed by the f-orbit of y. Moreover, there is a uniform bound (independent of x and y) for $\{D(G^n(\xi), F^n(\eta)) : n \in \mathbb{Z}\}$ that tends to 0 as g approaches f (see [**H1, L1**]).

It is known that for any $y \in M$ there exists $x \in M$ such that the g-orbit of x is shadowed by the f-orbit of y(see [**H1, L1**]).

Definition 2.4. — *Call J_g the set of $x \in M$ such that the g-orbit of x is shadowed by the f-orbit of some y.*

We remark that J_g consists of *all* points that are f-shadowed.

It is not difficult to see that $J_g = J_{g^n}$, if we consider f^n instead of f. J_g is a compact g-invariant set that not necessarily equals M. Moreover, there exists a continuous surjection homotopic to the inclusion, $h : J_g \to M$, such that $f \circ h = h \circ g|_{J_g}$ (see [**L1, H1**]). This implies that, as the equivalence classes on J_g coincide with $h^{-1}(y)$, $y \in M$, the quotient of J_g under the equivalence relation we are interested in, is homeomorphic to M and $\overline{g}$ restricted to this set is conjugated to f.

Assume now that g is Axiom A; we look for conditions in order to have semiconjugacy to f ($J_g = M$).

Given $\xi \in \mathcal{M}$, let

$$W_S^F(\xi) = \{\eta \in \mathcal{M}; D(F^n(\xi), F^n(\eta)) \to 0 \; as \; n \to +\infty\}$$
$$W_U^F(\xi) = \{\eta \in \mathcal{M}; D(F^n(\xi), F^n(\eta)) \to 0 \; as \; n \to -\infty\}$$

denote the F-stable and unstable sets of ξ.

Analogously, denote by $W_S^G(\xi)$ and $W_U^G(\xi)$ the G-stable and unstable manifolds of ξ.

When g is an Axiom A diffeomorphism, for $\xi \in \mathcal{M}$, B a basic piece of g and $\pi(\xi) \in B$, we shall denote

$${}^B W_S^G(\xi) = \{\eta \in W_S^G(\xi); \pi(\eta) \in B\}$$
$${}^B W_U^G(\xi) = \{\eta \in W_U^G(\xi); \pi(\eta) \in B\}.$$

Definition 2.5. — *We shall say that B is "small" iff ${}^B W_S^G(\xi)$ is bounded for some (and then for every) $\xi \in \pi^{-1}(B)$. In case that ${}^B W_S^G(\xi)$ is unbounded we say that B is "large".*

Observe that B being small implies that the quotient of B under the equivalence relation is a periodic orbit for $\overline{g}$ (all the points in a topologically mixing component of B are equivalent).

In [**L2**] was proved that large basic pieces for g always exist in the case that g is C^0-near enough to f. The same proof works for g isotopic to f because it is sufficient to take $x \in J_g$ such that its g-orbit is shadowed by an f-orbit dense in the future. Then, the basic piece containing the ω-limit set of x ($\omega_g(x)$) is large.

We will say that a closed curve is essential if it is not nul-homotopic.