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SHELAH'S CONJECTURE AND JOHNSON'S THEOREM
[*after* Will Johnson]

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1. Introduction

Beginning in 2019, Will Johnson has a published a remarkable series of papers (Johnson, 2019a,b, 2020a,b,c, 2021a,b) that prove the "Shelah Conjecture" for the case of fields of finite dp-rank. The theorem is:

Theorem 1.1. *If a field K is dp-finite then K is finite, or algebraically closed, or real-closed, or admits a non-trivial henselian valuation.*

The first three cases (finite, algebraically closed, real-closed) are well-understood model theoretically. Combining Johnson's theorem with a result from Halevi, Hasson, and Jahnke (2019), one obtains a straightforward characterization of dp-finite fields and a classification of their complete first-order theories in terms of familiar algebraic properties. A field (or rather its complete theory) is "dp-finite" if it admits a certain notion of rank (or dimension), which takes finite values.

In this note I aim to give a short account of Johnson's work, and the surrounding literature. I will begin with the principal definitions, results, and conjectures in the subject, explain the relationships between the conjectures, including the reduction to the 'V-topology conjecture'. Finally I will sketch the main ideas of Johnson's proof.

Acknowledgement. — The majority of the results, definitions, and ideas are due to Will Johnson. There are many notable exceptions, and I have tried to provide reasonably complete references, but undoubtedly there will be omissions. My intention is that everything without an explicit reference is understood by the reader to be due to Johnson. My thanks to the participants of the reading group on this topic, organised by Franziska Jahnke, as part of the *Decidability, definability and computability in number theory* program at the MSRI; and to Will Johnson who provided a very helpful extended summary to the reading group. Further thanks to Franziska Jahnke, Arno Fehm, Tamara Servi, and Will Johnson for comments on an earlier version. However, all mistakes are my own. My sincere thanks to the organizers of the Séminaire Bourbaki for this invitation.

Remark 1.2 (Notational conventions). Fields will often be denoted by letters like K, F, L, usually suppressing the field structure (i.e. the addition, multiplication, etc.). By $\mathscr{K} = (K, \ldots)$ we denote an expansion of a field K. Usually an elementary extension or an ultrapower of a field K will be denoted K^*, although a saturated elementary extension (also known as a 'monster model') will be denoted $\mathbb{K}$. The set of prime numbers will be denoted $\mathbb{P}$. Ordered abelian groups are understood to be totally ordered.

2. Some model theory of fields and valued fields

There are many excellent references for introductions to valuations, valued fields, and the model theory of valued fields. For example: Engler and Prestel (2005), Jahnke (2018), and van den Dries (2014).

Definition 2.1. A **valued field** is a pair (K, v) of a field K and a **valuation** $v : K \twoheadrightarrow \Gamma_v \cup \{\infty\}$, where the **value group** Γ_v is an ordered abelian group, written additively, such that

(**i**) $v(x) = \infty \iff x = 0$,

(**ii**) $v(xy) = v(x) + v(y)$, and

(**iii**) $v(x + y) \geqslant \min\{v(x), v(y)\}$.

The **valuation ring** $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$ and the **valuation ideal** $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$ each determine the valuation, up to isomorphism of the value group (commuting with the valuations), since:

$$v(x) \leqslant v(y) \iff yx^{-1} \in \mathcal{O}_v \iff xy^{-1} \notin \mathfrak{m}_v,$$

for all $x, y \in K^\times$. There is also the **residue field** $k_v := \mathcal{O}_v/\mathfrak{m}_v$. We say v is trivial if $\Gamma_v = \{0\}$. We say (K, v) is equicharacteristic/equal characteristic if $\mathrm{char}(K) = \mathrm{char}(k_v)$, otherwise we say it is mixed characteristic $(0, p)$ if $\mathrm{char}(K) = 0$ and $\mathrm{char}(k_v) = p$.

Remark 2.2. Two valuations v, w are **equivalent** if $\mathcal{O}_v = \mathcal{O}_w$. As remarked above, this holds if and only if there is an isomorphism $\varphi_\Gamma : \Gamma_v \longrightarrow \Gamma_w$ such that $w = \varphi_\Gamma \circ v$. As an abuse of language and notation, we usually identify equivalent valuations.

Definition 2.3. (K, v) is **henselian** if one (equivalently, all) of the following hold(s):

(**i**) The valuation v has a unique extension to the algebraic closure of K.

(**ii**) The valuation v extends uniquely to each finite extension of K.

(iii) For all monic $f \in \mathscr{O}_v[X]$ and $a \in \mathscr{O}_v$, if $f(a) \in \mathfrak{m}_v$ and $f'(a) \notin \mathfrak{m}_v$, there exists a unique $a' \in a + \mathfrak{m}_v$ with $f(a') = 0$.

(iv) For all monic $f \in \mathscr{O}_v[X]$ and $a \in \mathscr{O}_v$ with $v(f(a)) > 2v(f'(a))$, there exists $a' \in \mathscr{O}_v$ with $f(a') = 0$ and $v(a - a') > v(f'(a))$.

(v) All polynomials $f \in X^{n+1} + X^n + \mathfrak{m}_v[X]^{<n}$ have a root in K.

We also say that v itself is henselian. A field K is **henselian** if it admits a non-trivial henselian valuation. A henselian valued field (K, v) (or the valuation v itself) is **defectless** (respectively **separably defectless**) if $[L : K] = (\Gamma_w : \Gamma_v) \cdot [k_w : k_v]$ for every finite (resp. finite separable) extension $(L, w)/(K, v)$.

Henselianity it related to completeness: if $\Gamma_v \cong \mathbb{Z}$ and K is complete with respect to the ultrametric induced by v, then (K, v) is henselian. Every henselian (K, v) of residue characteristic zero is defectless.

Example 2.4. Of course there are so many examples worth discussing at this point, but let me introduce a few key ones.

(i) (K, v_{triv}): any field K can be equipped with the trivial valuation, i.e. such that $\mathscr{O}_v = K$. The value group is $\{0\}$, the residue field is K, and the valuation is henselian.

(ii) $(\mathbb{Q}, v_p)$: for any prime number p there is the **p-adic** valuation on $\mathbb{Q}$, given by

$$v_p(x) := \begin{cases} \ell & \text{for } x = p^\ell m/n,\ p \nmid m, n \text{ and } \ell, m, n \in \mathbb{Z} \\ \infty & \text{for } x = 0. \end{cases}$$

The value group is $\mathbb{Z}$, the residue field is $\mathbb{F}_p$, and the valuation is not henselian. The p-adic valuations and the trivial valuation are the only valuations on $\mathbb{Q}$, by a theorem of Ostrowski.

(iii) $(\mathbb{C}, v)$: $\mathbb{C}$ admits a large family of non-trivial valuations. Each of these valuations has divisible value group, algebraically closed residue field (all characteristics are possible), and these valuations are henselian and defectless.

(iv) Algebraic fields of positive characteristic (for example $\mathbb{F}_p$ and $\mathbb{F}_p^{\mathrm{alg}}$) admit only the trivial valuation.

(v) $(\mathbb{Q}_p, v_p)$: the **field of p-adic numbers** is the completion of $\mathbb{Q}$ with respect to v_p (i.e. with respect to the absolute value associated to v_p). This completion $\mathbb{Q}_p$ inherits a field structure, and it admits a unique valuation (also denoted v_p) such that $\mathbb{Q}_p$ is complete with respect to v_p. The value group is $\mathbb{Z}$, the residue field is $\mathbb{F}_p$, and the valuation is henselian and defectless.

(vi) $(F((\Gamma)), v_t)$: for each ordered abelian group Γ and each field F we may form the **generalized power series field/Hahn series field**, which is

$$F((\Gamma)) := \Big\{ \sum_{\gamma\in\Gamma} a_\gamma t^\gamma \;\Big|\; a_\gamma \in F \text{ and } \{\gamma \mid a_\gamma \neq 0\} \text{ is well-ordered} \Big\},$$

with both addition and multiplication 'as you would expect', that is:

$$\sum_{\gamma\in\Gamma} a_\gamma t^\gamma + \sum_{\gamma\in\Gamma} b_\gamma t^\gamma = \sum_{\gamma\in\Gamma} (a_\gamma + b_\gamma) t^\gamma,$$

and

$$\sum_{\gamma\in\Gamma} a_\gamma t^\gamma \cdot \sum_{\gamma\in\Gamma} b_\gamma t^\gamma = \sum_{\gamma\in\Gamma} \big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \big) t^\gamma.$$

We define the **t-adic valuation**

$$v_t\big(\sum_\gamma a_\gamma t^\gamma\big) := \min\{\gamma \mid a_\gamma \neq 0\}$$

and $v_t(0) := \infty$. The value group is Γ, the residue field is F, and the valuation is henselian and defectless.

(vii) One very important family of examples is the family of local fields of positive characteristic: $(\mathbb{F}_q((t)), v_t) := (\mathbb{F}_q((\mathbb{Z})), v_t)$, for q a prime power.

Remark 2.5 (Coarsenings and refinements). Let v, w be two valuations on a field K. We say that v is a **coarsening** of w if $\mathcal{O}_v \supseteq \mathcal{O}_w$; in this case we also say that w is a **refinement** of v. This defines a partial order on the set of valuations on K (up to equivalence). In fact the valuations are directed, in that there is a join $v \vee w$ of two valuations, which is the finest common coarsening. The valuation ring of $v \vee w$ is $\mathcal{O}_{v\vee w} = \mathcal{O}_v\mathcal{O}_w$. Moreover the family of valuations coarser than a given one is totally ordered (so, in this sense, the valuations form a tree). The coarsest valuation is v_{triv}. Two valuations v, w are **dependent** if $v \vee w$ is non-trivial, and **independent** otherwise. This is an equivalence relation on the non-trivial valuations.

Remark 2.6. The coarsenings w of a valuation v on a field K (up to equivalence) correspond bijectively to the convex subgroups of Γ_v:

$$\begin{aligned} \{\Delta \trianglelefteq_{\mathrm{convex}} \Gamma_v\} &\longleftrightarrow \{w \supseteq v\} \\ \Delta &\longleftrightarrow [w : x \longmapsto v(x) + \Delta]. \end{aligned}$$

This is surjective because each coarsening $w \supseteq v$ is equivalent to a valuation with value group equal to a quotient of Γ_v by a convex subgroup.

Remark 2.7 (Valuation topology). Let (K, v) be a valued field. We define a field topology T_v on K by declaring a basis of neighbourhoods of 0 to be given by $a \cdot \mathcal{O}_v$, for $a \in K^\times$. Of course, one must check that this really does give a field topology: we will discuss this more later. In fact, two non-trivial valuations induce the same topology if and only if they are dependent.

Remark 2.8. Note that T_v is indiscrete if and only if v is trivial. Some prefer to think of the topology induced by the trivial valuation as the discrete topology: this corresponds to declaring instead the basis to be given by sets of the form $a \cdot \mathfrak{m}_v$. For non-trivial valuations, these two definitions coincide, but for v_{triv} one gets the indiscrete topology or the discrete topology. The reason I prefer the indiscrete topology is that it is the coarsest topology, and $K = \mathcal{O}_{v_{\text{triv}}}$ is the coarsest valuation ring.

Definition 2.9. We introduce several first-order languages.

— $\mathfrak{L}_{\text{oag}} = \{+, -, 0, \leqslant, \boldsymbol{\infty}\}$ is the language of ordered abelian groups (written additively) with an additional symbol $\boldsymbol{\infty}$. Interpretations will be the disjoint union $\Gamma \sqcup \{\infty\}$, where Γ is an ordered abelian group and ∞ (the interpretation of $\boldsymbol{\infty}$) is an additional absorbing element 'at infinity', i.e. $x + \infty = \infty$ and $x \leqslant \infty$, for all x.

— $\mathfrak{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ (we will often suppress field structure from notation).

— $\mathfrak{L}_{\text{vf}} = \mathfrak{L}_{\text{ring}} \cup \{\mathcal{O}\}$ where $\mathcal{O}$ is a unary relation interpreted in a valued field (K, v) by the valuation ring $\mathcal{O}_v$.

— $\mathfrak{L}_{\text{div}} = \mathfrak{L}_{\text{ring}} \cup \{|\}$, where $|$ is a binary relation interpreted in a valued field (K, v) by writing $x \mid y$ if and only if $v(x) \leqslant v(y)$.

— $\mathfrak{L}_{\text{vf}-3}$, which is a three-sorted language, with two sorts $\mathbf{K}, \mathbf{k}$ equipped with $\mathfrak{L}_{\text{ring}}$ and a sort $\boldsymbol{\Gamma}$ equipped with $\mathfrak{L}_{\text{oag}}$. There are two unary function symbols $\mathbf{val} : \mathbf{K} \longrightarrow \boldsymbol{\Gamma}$ and $\mathbf{res} : \mathbf{K} \longrightarrow \mathbf{k}$. In a valued field (K, v) we interpret $\mathbf{K}$ by K, $\mathbf{k}$ by the residue field k_v, and $\boldsymbol{\Gamma}$ by the value group Γ_v (with an extra element ∞). The function symbol $\mathbf{val}$ is interpreted by the valuation, and $\mathbf{res}$ is interpreted by the residue map, extended to have the domain K by mapping each $x \notin \mathcal{O}_v$ to 0.

— $\mathfrak{L}_{\text{Pas}} = \mathfrak{L}_{\text{vf}-3} \cup \{\text{ac}\}$ be the expansion of $\mathfrak{L}_{\text{vf}-3}$ by a unary function symbol $\mathbf{ac} : K \longrightarrow k$, interpreted by an angular component map (which is a group homomorphism $\text{ac} : K^\times \longrightarrow k^\times$, extending the residue map on $\mathcal{O}_v$).

Remark 2.10. The choice of language in which to study valued fields can be very important, for example when considering properties like quantifier elimination. We will mostly be interested in combinatorial properties of the class of definable sets, with no