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THE CLASSICAL TITS QUADRANGLES

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ABSTRACT. — We show that a genuine Tits quadrangle that is 5-sturdy and laser-sharp but not exceptional is uniquely determined by either a quadratic space over a field or a pseudo-quadratic module over a simple associative ring with involution. This completes the classification of 7-sturdy laser-sharp Tits polygons.

RÉSUMÉ (*Les quadrangles de Tits classiques*). — Nous démontrons qu'un quadrangle de Tits véritable, 5-robuste et laser-tranchant mais non exceptionnel est déterminé de manière unique soit par un espace quadratique sur un corps, soit par un module pseudo-quadratique sur un anneau associatif simple à involution. Cela complète la classification des polygones de Tits 7-robustes et laser-tranchants.

1. Introduction

We call a relation that is symmetric and anti-reflexive an *opposition relation* and we say that an opposition relation $\equiv$ on a set Ω is k*-plump* for some $k \geq 2$, if for every subset X of Ω, such that $|X| \leq k$, there exists $y \in \Omega$ such that

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$x \equiv y$ for all $x \in X$. An opposition relation $\equiv$ on a set Ω is defined to be *trivial*, if $x \equiv y$ for all unequal $x, y \in \Omega$.

A Tits n-gon (for some $n \geq 3$) is a bipartite graph Γ in which for each vertex v, the set Γ_v of vertices adjacent to v is endowed with a 2-plump opposition relation satisfying axioms that reduce to the axioms of a Moufang n-gon in the case that all the opposition relations are trivial. We say that Γ is k-plump for some $k > 2$, if all the local opposition relations are k-plump (rather than just 2-plump).

Moufang polygons were classified in [21]. This classification says, roughly, that Moufang polygons are the spherical buildings arising from an exceptional algebraic group of relative rank 2 (or variations in characteristic 2 or 3) or $SL_3(K)$ for some skew-field K or some other classical group involving a non-degenerate form of Witt index 2 (but of arbitrary dimension).

There is a simple construction that produces a Tits polygon $X_{\Delta,T}$ from a Moufang spherical building Δ of rank at $r \geq 2$ and a suitable Tits index T; this Tits polygon is, in fact, k-plump whenever the panels of Δ contain at least $k+1$ chambers. We call these the Tits polygons *of index type*. If $r = 2$, then $X_{\Delta,T} = \Delta$, but if $r \geq 3$, then $X_{\Delta,T}$ is genuine, by which we mean it is a Tits polygon not all of whose opposition relations are trivial. The "projective planes" defined over a split octonion ring studied in [16] and [18] are examples of Tits triangles arising from this construction.

For every irreducible spherical building Δ of rank at least 3, there exists at least one suitable Tits index. Thus the theory of Tits polygons allows us to regard an arbitrary irreducible spherical building of rank at least 3 as a rank 2 structure to which the methods developed in [21] can be applied.

With a few exceptions, Tits polygons of index type satisfy a condition we call *dagger-sharp*. This is a natural condition on the action of the stabilizer of an apartment on the corresponding root groups. It is trivially satisfied by all Moufang polygons. Tits n-gons exist for every value of n, but we could show that dagger-sharp Tits n-gons exist only for $n = 3$, 4, 6 and 8. This result suggested the possibility of classifying dagger-sharp Tits polygons.

In [7], [8], [10] and [13], we classified the dagger-sharp Tits triangles, hexagons and octagons that are 7-plump. They are all either Moufang polygons or Tits polygons of index type or Tits triangles defined over a simple associative ring of infinite dimension over its center. This leaves only the Tits quadrangles.

As with Moufang n-gons, there is the greatest variety of examples of Tits n-gons when $n = 4$. The classification problem for $n = 4$ can be divided into several cases and in [9], [11], [12] and [14], we treated some of them. In this paper, we treat the remaining cases.

In the study of Tits quadrangles, we have found it necessary to strengthen just slightly the notions of k-plump and dagger-sharp to notions that we call k-sturdy and laser-sharp. In 11.8, we observe that every Tits polygon conforms to a root system Φ in the sense of 11.6. The k-sturdy condition is vacuous and laser-sharp is the same as dagger-sharp, except when $n = 4$ and either $\Phi = \mathrm{BC}_2$ or Φ is the root system BF_2 defined in 11.4. In these cases, there are roots α of Φ such that either 2α or $\sqrt{2}\alpha$ lies in Φ. For each such root α, it is natural to consider the subgroup of the root group U_α corresponding to 2α or $\sqrt{2}\alpha$ as an additional root group of Γ. The k-sturdy and laser-sharp conditions that go beyond k-plump and dagger-sharp are conditions on these subgroups.

When $\Phi = \mathrm{BC}_2$ or BF_2, we can distinguish two subcases. In [9], we classified the laser-sharp 4-sturdy Tits quadrangles in the first of these two subcases. The Tits quadrangles that arise are all associated with exceptional groups (and both $\Phi = \mathrm{BC}_2$ and $\Phi = \mathrm{BF}_2$ occur). In this paper, we show that in the second subcase, these quadrangles are all classical. By this we mean that they are parametrized (and uniquely determined) by a non-degenerate pseudo-quadratic module defined over a simple associative ring with involution (and $\Phi = \mathrm{BC}_2$). Combined with our earlier results, this result completes the classification of 7-sturdy laser-sharp Tits polygons. (See, however, 10.11 and 10.12.)

We remark that the dagger-sharp condition (or some version of this condition) is essential to our classification. There are many families of Tits polygons that are not dagger-sharp. In [8], we proved results about arbitrary Tits n-gons for $n = 3$, but little is known about arbitrary Tits n-gons for $n \geq 4$, and we regard their classification as a difficult problem.

In contrast, we do not believe our k-plump and k-sturdy conditions to be essential. The need for these conditions arises in many different ways, however (see, for example, 2.19 and 2.20), so it would not be easy to get along without them.

This paper is organized as follows. In Section 2, we introduce Tits polygons. In Section 3, we describe the three families of classical Tits quadrangles. In Section 4, we map out the basic strategy of first classifying the reduced Tits quadrangles (under the necessary plump and sharp conditions) and then classifying the remaining classical Tits quadrangles as extensions of the reduced ones. The exact result for reduced Tits quadrangles is formulated in Section 5 and proved in Sections 6–8. These Tits quadrangles are all classical. In Section 9, we classify the remaining classical Tits quadrangles. In Section 10, we say a few words about the classical Tits quadrangles of index type. In Section 11, finally, we give a survey of all the results that together yield the classification of 7-sturdy laser-sharp Tits polygons.

To conclude this Introduction, we say a few words about the larger context surrounding our classification. Let Φ be an irreducible finite root system of rank $r \geq 2$. When $r = 2$, we allow Φ to be the set of vertices of a regular $2m$-gon centered at the origin for any $m \geq 3$. Let Σ be a basis of Φ and let G be a group having a system of non-trivial subgroups $(U_\alpha)_{\alpha\in\Phi}$ satisfying the following conditions:

(MB1) If $\beta \in \Phi\backslash\{\alpha, \alpha'\}$, the commutator $[U_\alpha, U_\beta]$ is contained in the group generated by all U_γ, where γ runs through all roots that are linear combinations of α and β with strictly positive coefficients.

(MB2) For all non-trivial elements u of U_α, there is an element $m(u) \in U_{-\alpha} \cdot u \cdot U_{-\alpha}$ that, for each root β, conjugates U_β to $U_{r_\alpha(\beta)}$, where r_α is the reflection of the root system Φ corresponding to α.

(MB3) If $\delta \in \Sigma$, then $U_{-\delta}$ is not contained in $U_+ := \langle U_\beta \mid \beta \in \Sigma \rangle$.

(MB4) $G = \langle H, U_\alpha \mid \alpha \in \Phi \rangle$, where H the intersection of all the normalizers $N_G(U_\beta)$ with $\beta \in \Phi$.

As observed in [21, 42.3], the notion of a system of subgroups $(U_\alpha)_{\alpha\in\Phi}$ satisfying these four conditions is equivalent to the notion of an irreducible spherical building of rank r satisfying the Moufang condition. The notion of a Tits polygon "conforming to a root system" is just a reformulation of the axiom (MB1).

We consider two further axioms:

(MB2′) For *some* non-trivial element u of U_α, there is an element

$$m(u) \in U_{-\alpha} \cdot u \cdot U_{-\alpha},$$

which, for each root β, conjugates U_β to $U_{r_\alpha(\beta)}$, where r_α is as in (MB2).

(MB5) For each $\alpha \in \Phi$, $\langle U_\alpha, U_{-\alpha} \rangle = U_{-\alpha}U_\alpha U_{-\alpha}H_\alpha$, where $H_\alpha = H \cap \langle U_\alpha, U_{-\alpha} \rangle$.

A system of subgroups $(U_\alpha)_{\alpha\in\Phi}$ satisfying the axioms (MB1), (MB2′), (MB3) and (MB4) define a "root group grading of rank r" of the group G, a notion inspired by an analogous notion for Lie algebras. Root group gradings of groups have been studied, for example, in [15] and [26], but only in cases where the rank r of Φ is at least 3. When $r = 2$, these axioms seem to be too weak. We call such a system *stable* if, in addition to (MB1), (MB2′), (MB3) and (MB4), also (MB5) holds. By [7, 5.1–5.3], the notion of a group with a stable root group grading of rank 2 is, in fact, equivalent to the notion of a Tits polygon.

By a parametrization of a Tits polygon, we mean a pair $(\Lambda, (x_\alpha)_{\alpha\in\Phi})$, where Λ is an algebraic structure (a quadratic space (K, L, q), for example), and for each $\alpha \in \Phi$, x_α is an isomorphism from some part of Λ (from the additive group of K or the additive group of L in our example) to U_α such that the system $(x_\alpha)_{\alpha\in\Phi}$ is compatible with the axioms (MB1) and (MB2′). Thus our notion of a parametrization of a Tits polygon is a natural generalization of the notion of a pinning of a reductive group.

In [21], the classification of Moufang polygons is described in terms of such parametrizations. In each case, the corresponding algebraic data (a composition algebra, a pseudo-quadratic space, a quadratic Jordan algebra of degree 3, etc.) is required to be anisotropic in a suitable sense. It was always clear that the isotropic versions of these commutator relations should also be interesting, but it was only with the introduction of the notion of a Tits polygon that they could be given a geometric meaning.

Tits polygons have been applied to investigate the structure of absolutely simple algebraic groups of relative rank 1. In fact, this was our main goal when we introduced Tits polygons [13, Part III]. Let G, for instance, be a group of k-rational points of an absolutely simple algebraic group (for some field k) whose Tits index is one of the two Tits indices of absolute type E_8 and relative rank 1. The anisotropic part of such a Tits index is either D_7 or E_7. After replacing G by G modulo its center, the group G, it can be shown, arises from a Galois involution acting on a Tits quadrangle parametrized by a quadrangular algebra in the first case and a Tits hexagon parametrized by a Jordan algebra in the second, and in both cases, it is possible to use the parametrizing algebra to derive explicit formulas for the structure equations of the group G (as defined in [17, (3.1)]). This same goal was achieved with very different methods in [2] using structurable algebras, but with the restriction that the characteristic is not 2. Our calculations give hints of non-associative algebraic systems that can be used to parametrize all groups of relative rank 1 without this restriction.

Tits polygons are also closely related to the notion of “ring geometry” (as discussed, for example, in [23]). The spherical buildings associated with the k-rational points of an absolutely simple group whose Tits index has absolute type E_6, relative rank 2 and anisotropic part D_4 are Moufang triangles (or equivalently, projective planes) parametrized by an octonion division algebra. If we replace the octonion division algebra by a split octonion algebra in the commutator relations, we obtain a Tits triangle. In [18], Tits used these “projective planes” to classify the real forms of E_6 much as we used Tits quadrangles and hexagons to study rank 1 groups.

In [22], Veldkamp investigated to what extent the canonical construction of a projective plane from a three-dimensional right vector space over a skew-field K gives rise to an interesting geometrical object when K is replaced by an arbitrary associative ring. In the course of his investigations, Veldkamp discovered a connection to the notion of “stable range” introduced by Bass [1], a concept central to algebraic K-theory. Veldkamp’s work was extended by Faulkner (see [3]), but neither Veldkamp nor Faulkner seems to have been aware that the notion of a 2-plump opposition relation was hidden in their results. In fact, their results could be re-worked (in [8]) to yield the conclusion that an arbitrary 4-plump Tits triangle can be parametrized by a ring R that is alternative and has stable range 1 and that for each alternative ring R of stable range 1, there exists a unique Tits triangle parametrized by R. In [5], an analogous result for Tits quadrangles of pseudo-quadratic type is proved. The proof seems to yield new insights into K-theoretic questions and reveals new connections between unitary versions of the stable range condition and 2-plump opposition relations. This connection between these purely algebraic and purely geometric notions needs to be investigated further.