Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 151
Fascicule 2

2023

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 171-193

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 2, tome 151, juin 2023

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
151 (2), 2023, p. 171-193

FROM EXPONENTIAL COUNTING TO PAIR CORRELATIONS

by Jouni Parkkonen & Frédéric Paulin

Abstract. — We prove an abstract result on the correlations of pairs of elements in an exponentially growing discrete subset $\mathscr{E}$ of $[0,+\infty[$ endowed with a weight function. Assume that there exist $\alpha \in \mathbb{R}$, $c, \delta > 0$ such that, as $t \to +\infty$, the weighted number $\widetilde{\omega}(t)$ of elements of $\mathscr{E}$ that are not greater than t is equivalent to $c\,t^{\alpha} e^{\delta t}$. We prove that the distribution function of the differences of elements of $\mathscr{E}$ is $t \mapsto \frac{\delta}{2}\, e^{-|t|}$, and that, under an error term assumption on $\widetilde{\omega}(t)$, the pair correlation with a scaling with polynomial growth exhibits a Poissonian behaviour. We apply this result to answer a question of Pollicott and Sharp on the pair correlations of lengths of closed geodesics and common perpendiculars in negatively curved manifolds and metric graphs.

Texte reçu le 28 janvier 2022, modifié le 11 novembre 2022, accepté le 23 novembre 2022.

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Mathematical subject classification (2010). — 05A16, 11N45, 28A33, 37C35, 53C22; 26E99.

Key words and phrases. — Pair correlation, counting function, growth function, equidistribution, closed geodesics, common perpendiculars.

This research was supported by the French–Finnish CNRS IEA BARP.

doi:10.24033/bsmf.2867

Résumé (*Du comptage exponentiel aux corrélations de paires*). — Nous montrons un résultat abstrait sur la corrélation des paires d'éléments dans une partie $\mathscr{E}$ of $[0,+\infty[$ discrète, croissant exponentiellement et munie d'une fonction de poids. Supposons qu'il existe $\alpha \in \mathbb{R}$ et $c, \delta > 0$ tels que, quand $t \to +\infty$, le nombre pondéré $\widetilde{\omega}(t)$ d'éléments de $\mathscr{E}$ inférieurs à t soit équivalent à $c\,t^{\alpha} e^{\delta t}$. Nous montrons que la fonction de répartition des différences d'éléments de $\mathscr{E}$ est $t \mapsto \frac{\delta}{2}\, e^{-|t|}$, et que, sous condition d'existence d'un terme d'erreur sur $\widetilde{\omega}(t)$, la corrélation des paires pour un changement d'échelle à croissance au plus polynomiale admet un comportement poissonien. Nous utilisons ce résultat pour répondre à une question de Pollicott et Sharp sur la corrélation des paires de longueurs de géodésiques fermées et de perpendiculaires communes dans des variétés à courbure strictement négative et dans des graphes métriques.

1. Introduction

When studying the asymptotic distribution of a sequence of finite subsets of $\mathbb{R}$, finer information is sometimes given by the statistics of the spacing between pairs or k-tuples of elements, seen at an appropriate scaling. This problematic is largely developed in quantum chaos, including energy level spacings or clusterings, and in statistical physics, including molecular repulsion or interstitial distribution. See, for instance, [16, 2, 27, 3, 14, 12, 9]. In [25, 26], Pollicott–Sharp study the pair correlations of lengths of closed geodesics in negatively curved manifolds as the word length of the elements of the fundamental group that represent them tends to $+\infty$. They mention that a result replacing the word length by the Riemannian length does not seem to be available. One aim of this note is to give an answer to this problem, by a very general method.

For any set $\mathscr{E}$, a *weight function* (or multiplicity function when its values are positive integers) on $\mathscr{E}$ is simply a function $\omega : \mathscr{E} \to\,]0,+\infty[$. The *growth function* (or counting function when the weights are integers) of a locally finite subset $\mathscr{E}$ of $[0,+\infty[$ endowed with a weight function ω is the map $\mathscr{N}_{\mathscr{E},\omega} : [0,+\infty[\, \to [0,+\infty[$ defined by

$$\mathscr{N}_{\mathscr{E},\omega} : t \mapsto \sum_{x\in\mathscr{E}\cap[0,t]} \omega(x)\ .$$

Let $\mathscr{F} = (\,(F_N)_{N\in\mathbb{N}},\ \omega)$ be a nondecreasing sequence of finite subsets F_N of a finite dimensional Euclidean space E, endowed with a weight function $\omega : \bigcup_{N\in\mathbb{N}} F_N \to\,]0,+\infty[$. Let ψ be any function from $\mathbb{N}$ to $[1,+\infty[$, called a *scaling function*, and let $\psi' : \mathbb{N} \to\,]0,+\infty[$ be an appropriately chosen function, called a *renormalising function.* The *pair correlation measure of $\mathscr{F}$ at time N with scaling $\psi(N)$* is the measure on E with finite support

$$\mathscr{R}_N^{\mathscr{F},\psi} = \sum_{x,y\in F_N} \omega(x)\,\omega(y)\,\Delta_{\psi(N)(y-x)}\ , \tag{1}$$

where Δ_z denotes the unit Dirac mass at z in any measurable space. When the sequence of measures $\mathscr{R}_N^{\mathscr{F},\psi}$, renormalised by $\psi'(N)$, weak-star converges to a measure $g_{\mathscr{F},\psi}\,\mathrm{Leb}_E$ absolutely continuous with respect to the Lebesgue measure Leb_E of E, the Radon–Nikodym derivative $g_{\mathscr{F},\psi}$ is called the asymptotic *pair correlation function* of $\mathscr{F}$ *for the scaling* ψ *and renormalisation* ψ'. When $g_{\mathscr{F},\psi}$ is a positive constant, we say that $\mathscr{F}$ has a *Poissonian behaviour for the scaling* ψ *and renormalisation* ψ'.

Theorem 1.1. — *Let $\mathscr{E}$ be a locally finite subset of $[0,+\infty[$ endowed with a weight function ω. Assume that there exist $\alpha\in\mathbb{R}$, $c,\delta>0$ and $\kappa\geq 0$ such that, as $t\to+\infty$, we have*

$$\mathscr{N}_{\mathscr{E},\,\omega}(t)=c\,t^{\alpha}\,e^{\delta\,t}(1+o(e^{-\kappa\,t}))\ .$$

Let $\psi:\mathbb{N}\to[1+\infty[$ be an at most polynomially growing scaling function, with renormalising function $\psi':N\mapsto\frac{\mathscr{N}_{\mathscr{E},\,\omega}(N)^2}{\psi(N)}$. Then the weighted family $\mathscr{F}=((F_N=\{x\in\mathscr{E}:x\leq N\})_{N\in\mathbb{N}},\ \omega)$ admits a pair correlation function $g_{\mathscr{F},1}:t\mapsto\frac{\delta}{2}\,e^{-\delta\,|t|}$ if $\psi=1$ and has Poissonian behaviour with $g_{\mathscr{F},\psi}=\frac{\delta}{2}$ if $\lim\limits_{+\infty}\psi=\infty$ and $\kappa>0$.

We give some comments on the above statement at the beginning of Section 3. We refer to Theorem 3.1 for a more precise version, including error terms. The work on error terms constitutes the main technical parts of this paper.

Numerous settings in number theory, in geometry and in dynamical systems[1] give rise to counting functions that satisfy the assumption of Theorem 1.1. We will give some applications of the above result on geometry and dynamics in Section 4. Following the notation of [25], for all $a<b$ in $\mathbb{R}$ and $N\in\mathbb{N}$, let

$$\pi_{\mathscr{E}}(N,[a,b])=\mathscr{R}_N^{\mathscr{F},1}([a,b])=\sum_{x,y\in\mathscr{E}\,:\,x,y\leq N,\ a\leq y-x\leq b}\omega(x)\,\omega(y)$$

be the weighted number of differences of elements in $\mathscr{E}\cap[0,N]$ that lie in the interval $[a,b]$. Since the limit measure is atomless, under the assumptions of Theorem 1.1, we have the following corollary (see also Corollary 4.1).

Corollary 1.2. — *For all $a<b$ in $\mathbb{R}$, as $N\to+\infty$, we have*

$$\pi_{\mathscr{E}}(N,[a,b])\sim\frac{\delta}{2}\,\mathscr{N}_{\mathscr{E},\,\omega}(N)^2\int_a^b e^{-\delta|t|}\,dt\ .\quad\square$$

This answers the question of Pollicott–Sharp [25] when $\mathscr{E}$ is the set of lengths of closed geodesics in a closed negatively curved manifold, δ is the topological entropy of its geodesic flow and ω is the multiplicity function of these lengths (see Remark (3) in Section 3).

1. See, for instance [22], [7].

We suspect that when the scaling function has superexponential growth, the empirical measures $\frac{1}{\psi'(N)}\mathscr{R}_N^{\mathscr{F},\psi}$ have a total loss of mass as $N \to +\infty$ whatever the renormalising function ψ' is, and, hence, that the pair correlation function $g_{\mathscr{F},\psi}$ exists and is identically 0. The main open problem related to Theorem 1.1 is to study the pair correlations for scaling functions ψ, which are at the threshold, that is, are just exponentially growing. For instance, the set $\mathscr{E} = \{\ln n : n \in \mathbb{N} - \{0\}\}$ endowed with the trivial multiplicity function $\omega : x \mapsto 1$ satisfies the assumption of the above theorem with $c = 1$, $\alpha = 0$ and $\delta = 1$, thus recovering the case $\alpha = 0$ of [20, Theo. 1.1]. In [20], we study the pair correlations of this family for general scalings and some arithmetic weights functions, proving surprising level repulsion phenomena when $\psi(N) = e^N$. See also [21] for a study of the pair correlations of logarithms of complex lattice points at various scalings, where taking a Hausdorff–Gromov limit of the underlying space turns out to be necessary, and also yields, at the threshold scaling, an exotic asymptotic correlation function with a level repulsion phenomena.

2. Preliminaries on the growth of positive sequences

In this section, we recall some standard terminology used in the paper and we prove two technical results used in the proof of the main results in Section 3.

Recall that, given a set of parameters P and a positive map h defined on a neighbourhood of $+\infty$ in $\mathbb{N}$ or $\mathbb{R}$, we denote by $\mathrm{O}_P(h)$ (respectively, $\mathrm{o}_P(h)$) any *Landau function* (as the variable goes to $+\infty$) from $\mathbb{R}$ to $\mathbb{R}$, such that there exists a constant $M > 0$ depending only on the parameters in P and $t_0 \geq 0$ possibly depending on ambient data, such that for every $t \geq t_0$, we have $|\mathrm{O}_P(h)(t)| \leq M\, h(t)$ (respectively, such that $\lim\limits_{+\infty} \frac{|\mathrm{o}_P(h)(t)|}{h(t)} = 0$).

A positive sequence $(x_n)_{n\in\mathbb{N}}$ is

- *subexponentially growing* if for every $\gamma > 0$, we have $\lim_{n\to+\infty} \frac{x_n}{e^{\gamma n}} = 0$,
- *at most polynomially growing* if there exists $\gamma > 0$ such that $\lim_{n\to+\infty} \frac{x_n}{n^\gamma} = 0$,
- *strictly sublinearly growing with exponent* $\gamma \in\,]0,1[$ if $\lim_{n\to+\infty} \frac{x_n}{n^\gamma} = 0$.

Note that a constant positive sequence is strictly sublinearly growing with exponent any element of $]0,1[$.

The first result generalises a classical result on the geometric sums (when $b = 0$) to the generality needed for the proofs in Section 3.

Lemma 2.1. — *For every $b \in \mathbb{R}$, for every sequence $(a_M)_{M\in\mathbb{N}}$ in $]1,+\infty[$ such that the sequence $(\frac{1}{\ln a_M})_{M\in\mathbb{N}}$ is strictly sublinearly growing with exponent $\gamma \in\,]0,1[$, for every element $\gamma^* \in\,]\gamma,1[$, as M tends to $+\infty$ in $\mathbb{N}$, we have*

$$\sum_{k=1}^{M} k^b (a_M)^k = \frac{a_M}{a_M - 1} M^b (a_M)^M \Big(1 + \mathrm{O}_b\Big(\frac{1}{M^{1-\gamma*}}\Big)\Big)\,.$$

Proof. — By the definition of a strictly sublinearly growing sequence with exponent γ, we have $\lim_{M\to+\infty} M^\gamma \ln a_M = +\infty$. In particular, $\ln a_M \geq M^{-\gamma}$ if M is large enough. As a preliminary remark, note that we have $n^b(a_M)^n = \mathrm{O}_b(M^b(a_M)^M)$ for every $n \in \{1, \dots, M\}$. This is immediate if $b \geq 0$ and follows when $b < 0$ by considering separately the case $n \geq \frac{M}{2}$ (in which case, we have $\frac{n^b(a_M)^n}{M^b(a_M)^M} \leq 2^{|b|}$) and $n \leq \frac{M}{2}$ (in which case, we have

$$\frac{n^b(a_M)^n}{M^b(a_M)^M} \leq M^{|b|}(a_M)^{-M/2} = e^{-\frac{M^{1-\gamma}}{2}\left(M^\gamma \ln a_M - 2|b|\frac{\ln M}{M^{1-\gamma}}\right)} ,$$

which converges to 0 as M tends to $+\infty$).

Recall that $(1-\frac{1}{n+1})^b = 1+\mathrm{O}_b(\frac{1}{n})$ for every $n \geq 1$. With $\Sigma_M = \sum_{n=1}^M n^b(a_M)^n$, for every $S \in [1, M]$, by the standard telescopic sum argument and by cutting the third sum below for $n \leq S$ and for $n > S$ (using in this second case the preliminary remark), we, hence, have

$$\begin{aligned}
(2) \quad (a_M - 1)\Sigma_M &= \sum_{n=1}^{M} n^b(a_M)^{n+1} - \Sigma_M \\
&= \sum_{n=1}^{M}(n+1)^b(1-\frac{1}{n+1})^b(a_M)^{n+1} - \Sigma_M \\
&= (M+1)^b(a_M)^{M+1} - a_M + \mathrm{O}_b\big(\sum_{n=1}^{M}\frac{1}{n}(n+1)^b(a_M)^{n+1}\big) \\
&= (M+1)^b(a_M)^{M+1} - a_M + \mathrm{O}_b\big((S+1)^{b+1}(a_M)^{S+1}\big) \\
&\quad + \mathrm{O}_b\big(\frac{M-S}{S}(M+1)^b(a_M)^{M+1}\big) .
\end{aligned}$$

As $M \to +\infty$, we take $S = M - M^{\gamma*}$, which is equivalent to M since $\gamma^* \in\,]0, 1[$. We have $M - S = M^{\gamma*}$ and $\frac{M-S}{S} \sim \frac{1}{M^{1-\gamma^*}}$. Furthermore, as $M \to +\infty$, we have

$$\frac{(S+1)^{b+1}(a_M)^{S+1}}{M^b(a_M)^{M+1}} = \mathrm{O}_b(e^{-(M-S)\ln a_M}) = \mathrm{O}_b(e^{-M^{\gamma^*}M^{-\gamma}}) = \mathrm{O}_b\left(\frac{1}{M^{1-\gamma^*}}\right) .$$

Hence, the sum of the last two $\mathrm{O}_b(\cdot)$ functions of Formula (2) is an $\mathrm{O}_b\big(\frac{M^b(a_M)^{M+1}}{M^{1-\gamma^*}}\big)$ function. The result follows. □

Let $(\mathscr{E}, \omega)$ be as in the statement of Theorem 1.1. In order to simplify the notation, let

$$(3) \qquad \widetilde{\omega} : t \mapsto \mathscr{N}_{\mathscr{E},\omega}(t) = \sum_{x\in\mathscr{E},\, x\leq t} \omega(x) .$$

This function is defined on $\mathbb{R}$ with the usual convention that a sum over an empty set of indices is 0. The local finiteness assumption of the subset $\mathscr{E}$ of