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Maison de la SMF
Case 916 - Luminy
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France
`commandes@smf.emath.fr`

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`



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# PARAMETRIC GEOMETRY OF NUMBERS OVER A NUMBER FIELD AND EXTENSION OF SCALARS

by Anthony Poëls & Damien Roy

*Dedicated to Jeff Thunder on his 60th birthday.*

Abstract. — The parametric geometry of numbers of Schmidt and Summerer deals with rational approximation to points in $\mathbb{R}^n$. We extend this theory to a number field $K$ and its completion $K_w$ at a place $w$ in order to treat approximation over $K$ to points in $K_w^n$. As a consequence, we find that exponents of approximation over $\mathbb{Q}$ in $\mathbb{R}^n$ have the same spectrum as their generalizations over $K$ in $K_w^n$. When $w$ has relative degree 1 over a place $\ell$ of $\mathbb{Q}$, we further relate approximation over $K$ to a point $\boldsymbol{\xi}$ in $K_w^n$, to approximation over $\mathbb{Q}$ to a point $\Xi$ in $\mathbb{Q}_\ell^{nd}$ obtained from $\boldsymbol{\xi}$ by extension of scalars, where $d$ is the degree of $K$ over $\mathbb{Q}$. By combination with a result of P. Bel, this allows us to construct algebraic curves in $\mathbb{R}^{3d}$ defined over $\mathbb{Q}$, of degree $2d$, containing points that are very singular with respect to rational approximation.

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Anthony Poëls, Bureau 116, Bâtiment Braconnier, Université Claude Bernard Lyon 1, 43, boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France • *E-mail :* `poels@math.univ-lyon1.fr`

Damien Roy, Département de Mathématiques, Université d'Ottawa, 150 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada • *E-mail :* `droy@uottawa.ca`

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Résumé (*Géométrie paramétrique des nombres sur un corps de nombres et extension des scalaires*). — La géométrie paramétrique des nombres de Schmidt et Summerer étudie l'approximation rationnelle des points de $\mathbb{R}^n$. Nous étendons cette théorie à un corps de nombres $K$ et à son complété $K_w$ en une place $w$ pour traiter de l'approximation sur $K$ des points de $K_w^n$. Nous en déduisons que les exposants d'approximation sur $\mathbb{Q}$ des points de $\mathbb{R}^n$ possèdent le même spectre que leurs généralisations sur $K$ dans $K_w^n$. Lorsque $w$ est de degré relatif égal à un au-dessus d'une place $\ell$ de $\mathbb{Q}$, nous relions aussi l'approximation sur $K$ d'un point $\boldsymbol{\xi}$ de $K_w^n$ à celle sur $\mathbb{Q}$ du point $\Xi$ de $\mathbb{Q}_\ell^{nd}$ obtenu à partir de $\boldsymbol{\xi}$ par extension des scalaires, où $d$ désigne le degré de $K$ sur $\mathbb{Q}$. En combinant cette observation à un résultat de P. Bel, nous parvenons ainsi à construire des courbes algébriques dans $\mathbb{R}^{3d}$ définies sur $\mathbb{Q}$, de degré $2d$, contenant des points qui sont très singuliers vis à vis de l'approximation rationnelle.

## 1. Introduction

In Diophantine approximation, one is interested in measuring how well a given non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ with $n \geq 2$ can be approximated by subspaces of $\mathbb{R}^n$ defined over $\mathbb{Q}$ of a given dimension $k$. The most important cases are $k = 1$ and $k = n - 1$, and each naturally gives rise to a pair of exponents of approximation. For $k = 1$, they are $\widehat{\lambda}(\boldsymbol{\xi})$ (resp. $\lambda(\boldsymbol{\xi})$) defined as the supremum of all real numbers $\lambda$ for which the inequalities

$$(1) \qquad \|\mathbf{x}\| \leq Q \quad \text{and} \quad \|\mathbf{x} \wedge \boldsymbol{\xi}\| \leq Q^{-\lambda}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for each large enough $Q \geq 1$ (resp. for arbitrarily large values of $Q \geq 1$). For $k = n-1$, they are $\widehat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) defined as the supremum of all real numbers $\omega$ for which the inequalities

$$(2) \qquad \|\mathbf{x}\| \leq Q \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \leq Q^{-\omega}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for each large enough $Q \geq 1$ (resp. for arbitrarily large values of $Q \geq 1$), where the dot represents the usual scalar product in $\mathbb{R}^n$. This is independent of the choice of norms in $\mathbb{R}^n$ and in $\bigwedge^2 \mathbb{R}^n$, but for convenience, we use the Euclidean norms. As these exponents depend only on the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(\mathbb{R})$, we may assume that $\|\boldsymbol{\xi}\| = 1$. We refer the reader to the paper by Laurent [9] for generalizations in intermediate dimensions $k$.

While studying such exponents, it is important to restrict ourselves to points $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates, as this yields simpler statements and can be achieved by dropping redundant coordinates if necessary. For such points, a result of Dirichlet gives

$$(n-1)^{-1} \leq \widehat{\lambda}(\boldsymbol{\xi}) \leq \lambda(\boldsymbol{\xi}) \quad \text{and} \quad n - 1 \leq \widehat{\omega}(\boldsymbol{\xi}) \leq \omega(\boldsymbol{\xi}).$$

However, this does not fully describe the *spectrum* of $(\widehat{\lambda}, \lambda, \widehat{\omega}, \omega)$, namely the set of all quadruples $(\widehat{\lambda}(\boldsymbol{\xi}), \lambda(\boldsymbol{\xi}), \widehat{\omega}(\boldsymbol{\xi}), \omega(\boldsymbol{\xi}))$ associated with these $\boldsymbol{\xi}$. For $n = 2$, a complete description is simply given by

$$1 = \widehat{\lambda}(\boldsymbol{\xi}) = \widehat{\omega}(\boldsymbol{\xi}) \le \lambda(\boldsymbol{\xi}) = \omega(\boldsymbol{\xi}) \le \infty$$

For $n = 3$, the description is more complicated and was achieved by Laurent in [8], showing it as a semi-algebraic set. One of the constraints that it involves is the following remarkable identity due to Jarník [7, Satz 1],

$$\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})} - 1 = \frac{1}{\widehat{\omega}(\boldsymbol{\xi}) - 1}, \tag{3}$$

which together with $2 \le \widehat{\omega}(\boldsymbol{\xi}) \le \infty$ fully describes the spectrum of the pair $(\widehat{\lambda}, \widehat{\omega})$. For $n \ge 4$, the spectrum of the four exponents is not known, but Marnat [11] has shown that it contains an open subset of $\mathbb{R}^4$, and thus obeys no algebraic relation such as (3).

Much of the recent progress, including the breakthrough of Marnat and Moshchevitin [12] who determined the spectra of the pairs $(\widehat{\lambda}, \lambda)$ and $(\widehat{\omega}, \omega)$ for each $n \ge 3$, use Schmidt's and Summerer's parametric geometry of numbers [22] in a crucial way. In the dual but equivalent setting of [17], this theory attaches to any point $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\|\boldsymbol{\xi}\| = 1$, the family of symmetric convex bodies of $\mathbb{R}^n$

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \{\mathbf{x} \in \mathbb{R}^n \,;\, \|\mathbf{x}\| \le 1 \text{ and } |\mathbf{x} \cdot \boldsymbol{\xi}| \le e^{-q}\} \subseteq \mathbb{R}^n$$

parametrized by real numbers $q \ge 0$. For each $j = 1, \dots, n$, let $L_{\boldsymbol{\xi},j}(q)$ denote the logarithm of the $j$-th minimum of $\mathcal{C}_{\boldsymbol{\xi}}(q)$ with respect to $\mathbb{Z}^n$, namely the smallest real number $t$ such that $e^t\mathcal{C}_{\boldsymbol{\xi}}(q)$ contains at least $j$ linearly independent points of $\mathbb{Z}^n$. Then, form the map

$$\begin{aligned} \mathbf{L}_{\boldsymbol{\xi}} \colon [0,\infty) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto (L_{\boldsymbol{\xi},1}(q), \dots, L_{\boldsymbol{\xi},n}(q)) \,. \end{aligned} \tag{4}$$

Transposed in this setting, the main results of Schmidt and Summerer in [22] can be summarized as follows. Firstly, they note that the standard exponents of approximation to $\boldsymbol{\xi}$, including the four mentioned above, are given by simple formulas in terms of the inferior and superior limits of the ratios $L_{\boldsymbol{\xi},j}(q)/q$ as $q$ goes to infinity. Secondly, they show the existence of a constant $\gamma \ge 0$ and of a continuous piecewise linear map $\mathbf{P} \colon [0,\infty) \to \mathbb{R}^n$ with growth conditions involving $\gamma$, such that the difference $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded. Thus, the above-mentioned exponents of approximation to $\boldsymbol{\xi}$ can be computed, via the same formulas, in terms of the behaviour of $\mathbf{P}$ at infinity. They call such a map $\mathbf{P}$ an $(n,\gamma)$-system, and their set increases as the deformation parameter $\gamma$ increases. The $(n,0)$-systems, whose simpler description is recalled in Section 2, are simply called $n$-systems for brevity.

The main result of [17] provides a converse and shows more precisely that the set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ with $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\|\boldsymbol{\xi}\| = 1$ coincides with the set of $n$-systems modulo the additive group of bounded functions from $[0,\infty)$ to $\mathbb{R}^n$.

Moreover, $\boldsymbol{\xi}$ has $\mathbb{Q}$-linearly independent coordinates if and only if any corresponding $n$-system $\mathbf{P} = (P_1, \dots, P_n)$ satisfies $\lim_{q\to\infty} P_1(q) = \infty$. This reduces the determination of the spectrum of a family of exponents of approximation to a combinatorial problem about such $n$-systems.

A similar theory is developed in [19], with $\mathbb{Q}$ replaced by a field of rational functions in one variable $F(T)$ over an arbitrary field $F$, and $\mathbb{R}$ replaced by the completion $F((1/T))$ of $F(T)$ for the degree valuation.

The first goal of this paper is to extend the theory to a number field $K$ and its completion $K_w$ at a place $w$, in order to study approximation over $K$ to an arbitrary non-zero point $\boldsymbol{\xi}$ of $K_w^n$. In the next section, we show how to attach to such a point a function $\mathbf{L}_{\boldsymbol{\xi}}\colon [0,\infty) \to \mathbb{R}^n$ from which the four exponents of approximation to $\boldsymbol{\xi}$ can be computed in the same way as in the case where $K$ is $\mathbb{Q}$, and $w$ is its place at infinity. We will show that this set of maps also coincides with the set of $n$-systems modulo bounded functions. Thus, the spectrum of these exponents remains the same in this new context. In particular, Jarník's identity (3) holds for any point $\boldsymbol{\xi}$ of $K_w^3$ with linearly independent coordinates over $K$.

The second goal of this paper deals with extension of scalars from $\mathbb{Q}$ to a number field $K$. For this, we assume that $w$ is a place of $K$ of relative degree 1 over $\mathbb{Q}$, so that $K_w = \mathbb{Q}_\ell$ for the place $\ell$ of $\mathbb{Q}$ induced by $w$. We also choose a basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of $K$ over $\mathbb{Q}$ and for each point $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in K_w^n$ we define

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1\boldsymbol{\xi}, \dots, \alpha_d\boldsymbol{\xi}) \in K_w^{nd} = \mathbb{Q}_\ell^{nd} \tag{5}$$

and say that $\Xi$ is obtained from $\boldsymbol{\xi}$ by *extending scalars from* $\mathbb{Q}$ *to* $K$. If $\boldsymbol{\xi}$ has linearly independent coordinates over $K$, then $\Xi$ has linearly independent coordinates over $\mathbb{Q}$, and we will show a close relationship between the maps $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{L}_\Xi$. From this, we will deduce formulas linking the Diophantine exponents of approximation to $\boldsymbol{\xi}$ over $K$ with those of $\Xi$ over $\mathbb{Q}$. As a consequence, we will see that Jarník's identity (3) yields

$$\frac{1}{\widehat{\lambda}(\Xi)} - (2d-1) = \frac{d^2}{\widehat{\omega}(\Xi) - (2d-1)}, \tag{6}$$

for any $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} \in \mathbb{Q}_\ell^{3d}$ constructed from a point $\boldsymbol{\xi} \in K_w^3$ with $K$-linearly independent coordinates.

Let $\ell$ be a place of $\mathbb{Q}$. We say that a point $\boldsymbol{\xi} \in \mathbb{Q}_\ell^n$ is *very singular* if it has linearly independent coordinates over $\mathbb{Q}$ and satisfies $\widehat{\lambda}(\boldsymbol{\xi}) > 1/(n-1)$. This requires $n \geq 3$. Moreover, by Schmidt's subspace theorem, such a point is not algebraic, and so it generates a field $\mathbb{Q}(\boldsymbol{\xi})$ of transcendence degree at least 1 over $\mathbb{Q}$. The third goal, and the initial motivation of this paper, is to provide new examples of very singular points of transcendence degree 1. Up to now, all known examples come from dimension $n = 3$ and, apart from the constructions of [16], they are all of the form $\boldsymbol{\xi} = (1, \xi, \xi^2)$. Moreover, the supremum of $\widehat{\lambda}(1,\xi,\xi^2)$ for a transcendental number $\xi \in \mathbb{Q}_\ell$ is $1/\gamma \simeq 0.618$, where $\gamma = (1+\sqrt{5})/2$ denotes the golden ratio. For $\mathbb{Q}_\ell = \mathbb{R}$, this follows from the constructions of [14, 15] together with the upper bound of [6, Theorem 1a]. For a prime number $\ell$, this follows from [25, Chapter 2] or [3] together with [23, Théorème 2]. More generally, in [1], Bel showed that the result extends to any number field $K$ and its completion $K_w$ at a place $w$. Assuming that $w$ extends $\ell$ with relative degree 1 and choosing a basis $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d)$ of $K$ over $\mathbb{Q}$, we will deduce that $\mathbb{Q}_\ell^{3d}$ contains very singular points of the form $(\boldsymbol{\alpha},\xi\boldsymbol{\alpha},\xi^2\boldsymbol{\alpha})$ with $\xi \in \mathbb{Q}_\ell$.

## 2. Notation and main results

Throughout this paper, we fix an algebraic extension $K$ of $\mathbb{Q}$ of finite degree $d$.

**2.1. Absolute values**. — We denote by $M(K)$ the set of non-trivial places of $K$ and by $M_\infty(K)$ the subset of its archimedean places. For each $v \in M(K)$, we denote by $K_v$ the completion of $K$ at $v$ and by $d_v = [K_v : \mathbb{Q}_v]$ its local degree. When $v \in M_\infty(K)$, we normalize the absolute value $|\ |_v$ on $K_v$ so that it extends the usual absolute value $|\ |_\infty$ on $\mathbb{Q}$. Then $K_v$ embeds isometrically into $\mathbb{C}$. We identify it with its image $\mathbb{R}$ or $\mathbb{C}$ and write $v \mid \infty$. Otherwise, there is a unique prime number $p$ with $|p|_v < 1$, and we ask that $|p|_v = p^{-1}$, so that $|\ |_v$ extends the usual $p$-adic absolute value on $\mathbb{Q}$. We then write $v \mid p$. For these normalizations and for each $a \in K^*$, the product formula reads

$$\prod_{v\in M(K)} |a|_v^{d_v/d} = 1.$$

**2.2. Local norms and heights**. — Given a positive integer $n$ and a place $v$ in $M(K)$, we define the norm of a point $\mathbf{x} = (x_1,\dots,x_n)$ in $K_v^n$ by

$$\|\mathbf{x}\|_v = \begin{cases} (|x_1|_v^2 + \cdots + |x_n|_v^2)^{1/2} & \text{if } v \mid \infty, \\ \max\{|x_1|_v,\dots,|x_n|_v\} & \text{otherwise.} \end{cases}$$

For this choice of local norms, we define the *height* of a non-zero point $\mathbf{x}$ in $K^n$ by

$$H(\mathbf{x}) = \prod_{v\in M(K)} \|\mathbf{x}\|_v^{d_v/d}.$$

By the product formula, it depends only on the class of $\mathbf{x}$ in $\mathbb{P}^{n-1}(K)$ and satisfies $H(\mathbf{x}) \geq 1$.

More generally, for each $k \in \{1,\dots,n\}$ and each $v \in M(K)$, we define the norm of a point in $\bigwedge^k K_v^n$ to be the norm of its set of Plücker coordinates in $K_v^N$, where $N = \binom{n}{k}$. We also define the height of a point in $\bigwedge^k K^n$ to be