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A SPECIAL DEBARRE–VOISIN FOURFOLD

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ABSTRACT. — Consider the finite simple group $\mathbf{G} \coloneqq \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660, which has an irreducible representation V_{10} of dimension 10. In this note, we study a special trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ that is $\mathbf{G}$-invariant. Following the construction of Debarre–Voisin, we obtain a smooth hyperkähler fourfold $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ with many symmetries. We will also look at the associated Peskine variety $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$, which is highly symmetric as well and admits 55 isolated singular points. It will help us to better understand the geometry of the special Debarre–Voisin fourfold $X_6^{\sigma_0}$. We also discuss an application of this example to the global geometry of the moduli space of Debarre–Voisin fourfolds.

RÉSUMÉ (*Une variété de Debarre–Voisin spéciale*). — Considérons le groupe simple fini $\mathbf{G} \coloneqq \mathrm{PSL}(2, \mathbf{F}_{11})$ d'ordre 660, qui admet une représentation irréductible V_{10} de dimension 10. Nous allons étudier un trivecteur $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ qui est $\mathbf{G}$-invariant. En suivant la construction de Debarre–Voisin, nous obtenons une variété hyperkählérienne $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ lisse de dimension 4 avec beaucoup de symétries. Nous allons aussi étudier la variété de Peskine associée $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$, qui admet 55 points singuliers isolés et est également très symétrique. Cette dernière nous permet de mieux comprendre la géométrie de la variété spéciale $X_6^{\sigma_0}$. Nous discuterons aussi d'une application de cet exemple à la géométrie globale de l'espace de modules des variétés de Debarre–Voisin.

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1. Introduction

The study of automorphism groups for K3 surfaces and higher dimensional hyperkähler manifolds is a rich subject that has many deep relations with lattice theory and representation theory of simple groups. For example, in [13], Mukai showed that a finite group of symplectic automorphisms of a K3 surface is always a subgroup of the Mathieu group M_{23}. Similarly, in [12, Theorem 7.2.4], Mongardi showed that a finite group of symplectic automorphisms of a hyperkähler manifold of $\mathrm{K3}^{[2]}$-type is a subgroup of the Conway group Co_1. Moreover, for a such manifold X, the maximal prime order of any symplectic automorphism is 11, and in this case, X must have maximal Picard rank 21, so it is isolated in the moduli.

Consider the finite simple group $\mathbf{G} \coloneqq \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660. Mongardi constructed a special cubic fourfold, as well as a special Eisenbud–Popescu–Walter sextic with a faithful $\mathbf{G}$-action. From these, one obtains two hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type—the corresponding Fano variety of lines and double EPW sextic—that are highly symmetric (see [12, Section 4.5] and [7]). We also note that a complete classification of symplectic automorphism groups for cubic fourfolds is available in [10].

In this paper, we study an explicit example of a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type in the Debarre–Voisin family that also admits a faithful $\mathbf{G}$-action. A key feature of this example is that we can describe explicitly its Picard lattice using the geometry of some associated Fano varieties.

Let V_{10} be a 10-dimensional complex vector space. A *Debarre–Voisin variety* X_6^σ is defined inside the Grassmannian $\mathrm{Gr}(6, V_{10})$ from the datum of a trivector $\sigma \in \bigwedge^3 V_{10}^\vee$. By studying the representations of the group $\mathbf{G}$, it is not hard to find a candidate for the special trivector σ_0: denote by V_{10} one of the two 10-dimensional irreducible representations of $\mathbf{G}$; there exists a unique (up to multiplication by a scalar) trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ that is $\mathbf{G}$-invariant.

Using the general results obtained in [2] on the geometry of Debarre–Voisin varieties and associated Peskine varieties, one can study in detail the geometry of this special Debarre–Voisin variety $X_6^{\sigma_0}$. We prove the following results.

THEOREM 1.1. — *Let $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ be the special $\mathbf{G}$-invariant trivector.*

1. *(Proposition 3.2) The Debarre–Voisin variety $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ is smooth of dimension* 4.
2. *(Proposition 4.2) The associated Peskine variety $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$ has* 55 *isolated singular points. The group $\mathbf{G}$ acts transitively on them.*
3. *(Proposition 5.6) The group $\mathrm{Aut}_H^s(X_6^{\sigma_0})$ of symplectic automorphisms that fix the polarization H on $X_6^{\sigma_0}$ is isomorphic to $\mathbf{G}$.*
4. *One can give an explicit description of the Picard lattice of $X_6^{\sigma_0}$, which has maximal rank* 21. *It is spanned by* 55 (-2)*-classes (see* (4) *for the*

Gram matrix). Moreover, if we denote by $H^2_{\mathrm{trans}}(X_6^{\sigma_0})$ *the transcendental lattice and by* $T := H^2(X_6^{\sigma_0}, \mathbf{Z})^{\mathbf{G}}$ *the* $\mathbf{G}$*-invariant sublattice, we have the following isomorphisms of lattices (Proposition 5.9)*

$$H^2_{\mathrm{trans}}(X_6^{\sigma_0}) \simeq L_{11} := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 6 \end{smallmatrix}\right), \qquad T = H^2_{\mathrm{trans}}(X_6^{\sigma_0}) \oplus \langle H \rangle \simeq L_{11} \oplus (22),$$
$$\mathrm{Pic}(X_6^{\sigma_0}) \simeq U \oplus E_8(-1)^{\oplus 2} \oplus L(-1),$$

where the component L *can be taken to be both* $\left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 8 \end{smallmatrix}\right)$ *and* $L_{11} \oplus (2)$*.*

5. *(Proposition 5.12)* $X_6^{\sigma_0}$ *can be characterized as the unique Debarre–Voisin fourfold admitting a symplectic automorphism of order* 11*.*
6. $X_6^{\sigma_0}$ *is birationally isomorphic to the Hilbert square of a K3 surface (Proposition 5.14) and is special in the sense of Hassett for all possible discriminants* $d \geq 24$ *(Proposition 5.15).*

The property of being Hassett special for all possible discriminants $d \geq 24$ has a nice implication on the global geometry of the moduli space of Debarre–Voisin fourfolds. Namely, we have two different moduli spaces in this setting: the GIT moduli space $\mathcal{M}_{\mathrm{DV}}$ of trivectors and the moduli space $\mathcal{M}_{22}^{(2)}$ of polarized hyperkähler manifolds. The Debarre–Voisin construction provides a rational map

$$\mathfrak{m} \colon \mathcal{M}_{\mathrm{DV}} \dashrightarrow \mathcal{M}_{22}^{(2)},$$

which is proved to be birational [16, Theorem 1.8]. Moreover, one can show that the restriction of $\mathfrak{m}$ to the open locus $\mathcal{M}_{\mathrm{DV}}^{\mathrm{sm}}$ of trivectors defining a smooth Debarre–Voisin fourfold is an open immersion (Proposition 6.3).

When we resolve the indeterminacies of this map, the image of each exceptional divisor is called *Hassett–Looijenga–Shah* (HLS) (see Definition 6.4), which reflects a difference between the GIT and the Baily–Borel compactifications. The result on $X_6^{\sigma_0}$ implies that all Heegner divisors $\mathcal{D}_d$ for $d \geq 24$ are not HLS (Corollary 6.5). Combined with the results of [5] and [15], one concludes that a Heegner divisor $\mathcal{D}_d$ is HLS if and only if $d \in \{2, 6, 8, 10, 18\}$. We discuss this in Section 6.

Notation. — We use σ to denote a trivector and σ_0 to denote the special $\mathbf{G}$-invariant trivector.

2. The special trivector

We first give the construction of the special trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$.

The finite simple group $\mathbf{G} := \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660 admits eight different irreducible complex representations: two of them are of dimension 5 and will be denoted by V_5 and $V_5^\vee$. They are the dual to each other.

A classical result is that the symmetric power $\mathrm{Sym}^3 V_5^\vee$—the space of cubic polynomials on V_5—admits an irreducible subrepresentation of dimension 1:

for a suitable choice of basis $(y_0, \ldots, y_4)$ of $V_5^\vee$, this corresponds to the Klein cubic with equation

$$y_0^2 y_1 + y_1^2 y_2 + y_2^2 y_3 + y_3^2 y_4 + y_4^2 y_0 \in \operatorname{Sym}^3 V_5^\vee.$$

In [1], Adler showed that the automorphism group of this smooth cubic is precisely the group $\mathbf{G}$.

The wedge product $\bigwedge^2 V_5$ gives another irreducible representation, of dimension 10, which is self-dual and will be denoted by V_{10}. We consider elements of $\bigwedge^3 V_{10}^\vee$. A computation of characters tells us that this representation of $\mathbf{G}$ also admits one irreducible subrepresentation of dimension 1, generated by a $\mathbf{G}$-invariant trivector σ_0. The characters of all eight irreducible representations of $\mathbf{G}$ as well as the character of $\bigwedge^3 V_{10}^\vee$ can be found in Section B, Table B.1. Note that the other irreducible representation V_{10}' of dimension 10 does not provide $\mathbf{G}$-invariant trivectors (see also Remark 5.13 on the uniqueness of the trivector σ_0).

We now give a concrete description of the special trivector σ_0 in terms of coordinates in a suitable basis. The subgroup $\mathbf{B}$ of $\mathbf{G}$ of upper triangular matrices can be generated by the elements

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$$

of the respective orders 11 and 5. Write $\zeta = e^{2\pi i/11}$ and $\rho\colon \mathbf{G} \to \mathrm{GL}(V_5^\vee)$ for the representation $V_5^\vee$. In a suitable basis $(y_0, \ldots, y_4)$ of $V_5^\vee$, the matrices of P and R are

$$\rho(P) = \begin{pmatrix} \zeta^1 & 0 & 0 & 0 & 0 \\ 0 & \zeta^9 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^5 \end{pmatrix} \quad \text{and} \quad \rho(R) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1}$$

Note that one can already identify the equation of the $\mathbf{G}$-invariant Klein cubic using only these two elements, instead of the whole group $\mathbf{G}$.

The elements $y_{ij} \coloneqq y_i \wedge y_j$ form a basis of $V_{10}^\vee$. In this basis, we see that P acts diagonally and R as a permutation (see Table 2.1; note that we have chosen a particular order in which the action of R is very simple). We may easily verify that the space of trivectors invariant under the action of P and R is of dimension 2 and is spanned by the $\mathbf{B}$-invariant trivectors

$$\begin{aligned} \sigma_1 \coloneqq{}& y_{01} \wedge y_{23} \wedge y_{02} + y_{12} \wedge y_{34} \wedge y_{13} + y_{23} \wedge y_{40} \wedge y_{24} \\ &+ y_{34} \wedge y_{01} \wedge y_{30} + y_{40} \wedge y_{12} \wedge y_{41}, \\ \sigma_2 \coloneqq{}& y_{01} \wedge y_{41} \wedge y_{24} + y_{12} \wedge y_{02} \wedge y_{30} + y_{23} \wedge y_{13} \wedge y_{41} \\ &+ y_{34} \wedge y_{24} \wedge y_{02} + y_{40} \wedge y_{30} \wedge y_{13}. \end{aligned}$$

TABLE 2.1. The action of P and R in the basis (y_{ij})

	y_{01}	y_{12}	y_{23}	y_{34}	y_{40}	y_{02}	y_{13}	y_{24}	y_{30}	y_{41}
Eigenvalues of $\bigwedge^2 \rho(P)$	ζ^{10}	ζ^2	ζ^7	ζ^8	ζ^6	ζ^5	ζ^1	ζ^9	ζ^4	ζ^3
Action of $\bigwedge^2 \rho(R)$	y_{12}	y_{23}	y_{34}	y_{40}	y_{01}	y_{13}	y_{24}	y_{30}	y_{41}	y_{02}

To identify the unique $\mathbf{G}$-invariant trivector, we must look at some elements in $\mathbf{G} \smallsetminus \mathbf{B}$. Since the explicit description for the representation V_5 is known [18], we will pick one such element and compute its matrix explicitly.

The group $\mathbf{G}$ admits a presentation with two generators a, b and relations $a^2 = b^3 = (ab)^{11} = [a, babab]^2 = 1$. We can take $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. One may check that $ab = P$ while $bbababab bababab b = R$. Matrices for $\rho(a)$ and $\rho(b)$ are provided by [18], so the representation is completely determined. Choose a suitable basis of $V_5^\vee$ consisting of eigenvectors of $\rho(ab) = \rho(P)$. In this basis, the matrices of P and R are as in (1). Since the element a does not lie in the subgroup $\mathbf{B}$, we use its matrix in this new basis to verify that the unique (up to multiplication by a scalar) $\mathbf{G}$-invariant trivector is $\sigma_0 := \sigma_1 + \sigma_2$.

From now on, we will rewrite the basis (y_{ij}) as $(x_0, \ldots, x_9)$ in the order chosen in Table 2.1, so the trivector σ_0 is given by

$$\sigma_0 = x_0 \wedge x_2 \wedge x_5 + x_1 \wedge x_3 \wedge x_6 + x_2 \wedge x_4 \wedge x_7 + x_3 \wedge x_0 \wedge x_8 + x_4 \wedge x_1 \wedge x_9 \\ + x_0 \wedge x_9 \wedge x_7 + x_1 \wedge x_5 \wedge x_8 + x_2 \wedge x_6 \wedge x_9 + x_3 \wedge x_7 \wedge x_5 + x_4 \wedge x_8 \wedge x_6,$$

or more succinctly,

$$\sigma_0 = [025] + [136] + [247] + [308] + [419] \\ + [097] + [158] + [269] + [375] + [486]. \tag{2}$$

We have, therefore, shown the following result.

PROPOSITION 2.1. — *Up to multiplication by a scalar, the trivector σ_0 in (2) is the unique $\mathbf{G}$-invariant trivector in $\bigwedge^3 V_{10}^\vee$, where V_{10} is the 10-dimensional irreducible $\mathbf{G}$-representation given in Table B.1.*

3. The Debarre–Voisin fourfold

The Debarre–Voisin variety associated with a non-zero trivector σ is the scheme

$$X_6^\sigma := \{[V_6] \in \mathrm{Gr}(6, V_{10}) \mid \sigma|_{V_6} = 0\}$$

in the Grassmannian $\mathrm{Gr}(6, V_{10})$ parametrizing those $[V_6]$ on which σ vanishes. Its expected dimension is 4. For σ general, it is shown in [8] that X_6^σ is a smooth