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THE GEOMETRIC SATAKE CORRESPONDENCE FOR RAMIFIED GROUPS

BY XINWEN ZHU

ABSTRACT. – We prove the geometric Satake isomorphism for a reductive group defined over $F = k((t))$, and split over a tamely ramified extension. As an application, we give a description of the nearby cycles on certain Shimura varieties via the Rapoport-Zink-Pappas local models.

RÉSUMÉ. – Nous démontrons l'isomorphisme de Satake géométrique pour un groupe réductif défini sur $F = k((t))$ et déployé sur une extension modérément ramifiée. Nous donnons comme application une description des cycles évanescents sur certaines variétés de Shimura via les modèles locaux de Rapoport-Zink-Pappas.

Introduction

The Satake isomorphism (for unramified groups) is the starting point of the Langlands duality. Let us first recall its statement. Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$ and residue field k, and let G be a connected unramified reductive group over F (e.g., $G = \mathrm{GL}_n$). Let $A \subset G$ be a maximal split torus of G, and W_0 be the Weyl group of (G, A). Let K be a hyperspecial subgroup of $G(F)$ containing $A(\mathcal{O})$ (e.g., $K = \mathrm{GL}_n(\mathcal{O})$). Then the classical Satake isomorphism describes the spherical Hecke algebra $\mathrm{Sph} = C_c(K \setminus G(F)/K)$, the algebra of compactly supported bi-K-invariant functions on $G(F)$ under convolution. Namely, there is an isomorphism of algebras

$$\mathrm{Sph} \simeq \mathbb{C}[\mathbb{X}_\bullet(A)]^{W_0},$$

where $\mathbb{X}_\bullet(A)$ is the coweight lattice of A, and $\mathbb{C}[\mathbb{X}_\bullet(A)]^{W_0}$ denotes the W_0-invariants of the group algebra of $\mathbb{X}_\bullet(A)$.

If F has positive characteristic $p > 0$, then the classical Satake correspondence has a vast enhancement. For simplicity, let us assume that G is split over F (for the general case, see Theorem A.12). Let us write $G = H \otimes_k F$ for some split group H over k so that $K = H(\mathcal{O})$. Let $\mathrm{Gr}_H = H(F)/H(\mathcal{O})$ be the affine Grassmannian of H. Choose ℓ a prime different from p, and let Sat_H be the category of $(K \otimes \bar{k})$-equivariant perverse sheaves with $\overline{\mathbb{Q}}_\ell$-coefficients on $\mathrm{Gr}_H \otimes \bar{k}$. Then this is a Tannakian category and there is an equivalence

$$\mathrm{Sat}_H \simeq \mathrm{Rep}(G^\vee_{\overline{\mathbb{Q}}_\ell}),$$

where $G^\vee_{\overline{\mathbb{Q}}_\ell}$ is the dual group of G and $\mathrm{Rep}(G^\vee_{\overline{\mathbb{Q}}_\ell})$ is the tensor category of algebraic representations of $G^\vee_{\overline{\mathbb{Q}}_\ell}$ (cf. [10, 19]).

There is also a version of Satake isomorphism for an arbitrary reductive group over F, as recently proved by Haines and Rostami (cf. [12])(1). Namely, let $\mathscr{B}(G)$ be the Bruhat-Tits building of G and $v \in \mathscr{B}(G)$ be a special vertex. Let $K_v \subset G(F)$ be the special parahoric subgroup of $G(F)$ corresponding to v. Let A be a maximal split F-torus of G such that $K_v \supset A(\mathcal{O})$, let M be the centralizer of A in G and $W_0 = N_G(A)/M$ be the Weyl group as before. Let M_1 be the unique parahoric subgroup of $M(F)$, and $\Lambda_M = M(F)/M_1$, which is a finitely generated Abelian group. Then

$$C_c(K_v\backslash G(F)/K_v) \simeq \mathbb{C}[\Lambda_M]^{W_0}. \tag{0.1}$$

More explicitly, suppose that G is quasi-split so that $M = T$ is a maximal torus. Then

$$\Lambda_M = (\mathbb{X}_\bullet(T)_I)^\sigma,$$

where I is the inertial group and σ is the Frobenius, and $(\mathbb{X}_\bullet(T)_I)^\sigma$ denotes the σ-invariants of the I-coinvariants of the group $\mathbb{X}_\bullet(T)$.

The goal of this paper is to provide a geometric version of the above isomorphism when F has positive characteristic p and the group G is quasi-split and splits over a *tamely* ramified extension. More precisely, let k be an algebraically closed field and let $\ell \neq \mathrm{char}\, k$ be a prime. Let G be a group over the local field $F = k((t))$ (so that G is quasi-split automatically), which is split over a tamely ramified extension. That is, there is a finite extension $\tilde{F}/F$ such that $G_{\tilde{F}}$ is split and $\mathrm{char}\, k \nmid [\tilde{F} : F]$. Let $v \in \mathscr{B}(G)$ be a special vertex in the building of G and let $\underline{G}_v$ be the parahoric group scheme over $\mathcal{O} = k[[t]]$ (in the sense of Bruhat-Tits), determined by v. We write LG for the loop space of G and $K_v = L^+\underline{G}_v$ for the jet space of $\underline{G}_v$. By definition, for any k-algebra R, $LG(R) = G(R\hat{\otimes}_k F)$ and $K_v(R) = \underline{G}_v(R\hat{\otimes}_k \mathcal{O})$. Let

$$\mathcal{F}\ell_v = LG/K_v$$

be the (twisted) affine flag variety(2), which is an ind-scheme over k. Let $\mathcal{P}_v = \mathcal{P}_{K_v}(\mathcal{F}\ell_v)$ be the category of K_v-equivariant perverse sheaf on $\mathcal{F}\ell_v$, with coefficients in $\overline{\mathbb{Q}}_\ell$. Let H be a split Chevalley group over $\mathbb{Z}$ such that $G \otimes_F F^s \simeq H \otimes F^s$, where F^s is a (fixed) separable closure of F. Then there is a natural action of $I = \mathrm{Gal}(F^s/F)$ on $H^\vee := H^\vee_{\overline{\mathbb{Q}}_\ell}$ (preserving a fixed pinning).

THEOREM 0.1. – *The category $\mathcal{P}_v$ has a natural tensor structure. In addition, as tensor categories, there is an equivalence*

$$\mathcal{RS} : \mathrm{Rep}((H^\vee)^I) \simeq \mathcal{P}_v,$$

(1) There is another version, known earlier, as in [6].
(2) One would call $\mathcal{F}\ell_v$ the affine Grassmannian of G. However, we reserve the name "affine Grassmannian" of G for another object, as defined in Definition A.2.

such that $\mathrm{H}^* \circ \mathcal{RS}$ *is isomorphic to the forgetful functor, where* H^* *is the hypercohomology functor.*

This theorem can be regarded as a categorification of (0.1) in the case when k is algebraically closed and the group splits over a tamely ramified extension of $k((\mathrm{t}))$. For the description of $(H^\vee)^I$ when H is absolutely simple and simply-connected, see § 4.

Let us point out the following remarkable facts when the group is ramified. First, the group $(H^\vee)^I$ is not necessarily connected as is shown in Remark (4.4). Second, it is well-known that if G is unramified over F, then all the hyperspecial subgroups of G are conjugate under $G_{\mathrm{ad}}(F)$ ([27, §2.5]), where G_{ad} is the adjoint group of G. However, this is no longer true for special parahoric of G if G is ramified. An example is given by the odd ramified unitary similitude group GU_{2m+1}. There are essentially two types of special parahorics of GU_{2m+1}, as given in (7.1). One of them has reductive quotient GO_{2m+1} (denoted by $\underline{G}_{v_0}$), and the other has reductive quotient GSp_{2m} (denoted $\underline{G}_{v_1}$). Accordingly, the geometry of the corresponding flag varieties $\mathcal{F}\ell_{v_0}$ and $\mathcal{F}\ell_{v_1}$ are very different, while $\mathcal{P}_{v_0} \simeq \mathcal{P}_{v_1}$. Indeed, their Schubert varieties (i.e., closures of K_{v_i}-orbits) are both parameterized by irreducible representations of GO_{2m+1}. Let $\mathcal{F}\ell_{v_0\bar{\mu}_{2m,1}}$ (resp. $\mathcal{F}\ell_{v_1\bar{\mu}_{2m,1}}$) be the Schubert variety in $\mathcal{F}\ell_{v_0}$ (resp. $\mathcal{F}\ell_{v_1}$) parameterized by the standard representation of GO_{2m+1}. Then it is shown in [31] that $\mathcal{F}\ell_{v_0\bar{\mu}_{2m,1}}$ is not Gorenstein, while in [25] that $\mathcal{F}\ell_{v_1\bar{\mu}_{2m,1}}$ is smooth. On the other hand, the intersection cohomology of both varieties gives the standard representation of GO_{2m+1}. In addition, the stalk cohomologies of both sheaves are the "same". See Theorem 0.3 below.

REMARK 0.1. – Instead of considering a special parahoric K_v of LG, one can begin with the special maximal "compact" K'_v, (i.e., $K'_v = L^+\underline{G}'_v$, where $\underline{G}'_v$ is the stabilizer group scheme of v as constructed by Bruhat-Tits), and consider the category of K'_v-equivariant perverse sheaves on LG/K'_v. However, from a geometric point of view, this is less natural since K'_v is not necessarily connected and the category of K'_v-equivariant perverse sheaves is complicated. In fact, we do not know how to relate this category to the Langlands dual group yet. In addition, when we discuss the Langlands parameters in Section 6, it is also more "correct" to consider K_v rather than K'_v.

The idea of the proof of the theorem is as follows. Using Gaitsgory's nearby cycle functor construction as in [8, 31], we construct a functor

$$\mathcal{Z} : \mathrm{Sat}_H \to \mathcal{P}_v,$$

which is a central functor in the sense of [2]. By standard arguments in the theory of Tannakian equivalence and the Mirkovic-Vilonen theorem, this already implies that $\mathcal{P}_v \simeq \mathrm{Rep}(\tilde{G}^\vee)$ for certain closed subgroup $\tilde{G}^\vee \subset H^\vee$. Then we identify $\tilde{G}^\vee$ with $(H^\vee)^I$ using the parametrization of the K_v-orbits on $\mathcal{F}\ell_v$.

REMARK 0.2. – (i) We believe that the same argument (maybe with small modifications) should work for groups split over wild ramified extensions. However, we have not checked this carefully.

(ii) Our approach is more inspired by [8] rather than [19]. However, it would be interesting to know whether there is the similar theory of MV-cycles in the ramified case. It seems that