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INTERMEDIATE JACOBIANS AND RATIONALITY OVER ARBITRARY FIELDS

BY Olivier BENOIST AND Olivier WITTENBERG

ABSTRACT. – We prove that a three-dimensional smooth complete intersection of two quadrics over a field k is k-rational if and only if it contains a line defined over k. To do so, we develop a theory of intermediate Jacobians for geometrically rational threefolds over arbitrary, not necessarily perfect, fields. As a consequence, we obtain the first examples of smooth projective varieties over a field k which have a k-point, and are rational over a purely inseparable field extension of k, but not over k.

RÉSUMÉ. – Nous démontrons qu'une intersection complète lisse de deux quadriques de dimension 3 sur un corps k est k-rationnelle si et seulement si elle contient une droite définie sur k. À cet effet, nous développons une théorie des jacobiennes intermédiaires pour les variétés géométriquement rationnelles de dimension 3 sur des corps quelconques, non nécessairement parfaits. Comme conséquence, nous obtenons les premiers exemples de variétés projectives lisses sur un corps k qui ont un k-point, et sont rationnelles sur une extension de corps purement inséparable de k, mais pas sur k.

Introduction

Let k be a field. A variety X of dimension n over k is said to be *k-rational* (resp. *k-unirational*, resp. *separably k-unirational*) if there exists a birational map (resp. a dominant rational map, resp. a dominant and separable rational map) $\mathbf{A}^n_k \dashrightarrow X$.

This article is devoted to studying the k-rationality of threefolds over k. Our main result answers positively a conjecture of Kuznetsov and Prokhorov.

THEOREM A (Theorem 4.7). – *Let $X \subset \mathbf{P}^5_k$ be a smooth complete intersection of two quadrics. Then X is k-rational if and only if it contains a line defined over k.*

The question of the validity of Theorem A goes back to Auel, Bernardara and Bolognesi [5, Question 5.3.2 (3)], who raised it when k is a rational function field in one variable over an algebraically closed field.

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Using the fact that varieties X as in Theorem A are separably k-unirational if and only if they have a k-point (see Theorem 4.8), we obtain new counterexamples to the Lüroth problem over non-closed fields.

THEOREM B (Theorem 4.14). – *For any algebraically closed field κ, there exists a three-dimensional smooth complete intersection of two quadrics $X \subset \mathbf{P}^5_{\kappa((t))}$ which is separably $\kappa((t))$-unirational, $\kappa((t^{\frac{1}{2}}))$-rational, but not $\kappa((t))$-rational.*

When κ has characteristic 2, Theorem B yields the first examples of smooth projective varieties over a field k which have a k-point and are rational over the perfect closure of k, but which are not k-rational (see Remarks 4.15 (iii) and (iv)).

Theorem A may be compared to the classical fact that a smooth quadric over k is k-rational if and only if it has a k-point. However, although it is easy to check that a smooth projective k-rational variety has a k-point, the fact that a k-rational three-dimensional smooth complete intersection of two quadrics X necessarily contains a k-line is highly non-trivial. To prove it, we rely on obstructions to the k-rationality of X arising from a study of its intermediate Jacobian.

Such obstructions go back to the seminal work of Clemens and Griffiths [15]: if a smooth projective threefold over $\mathbf{C}$ is $\mathbf{C}$-rational, then its intermediate Jacobian is isomorphic, as a principally polarized abelian variety over $\mathbf{C}$, to the Jacobian of a (not necessarily connected) smooth projective curve. This implication was used in [15] to show that smooth cubic threefolds over $\mathbf{C}$ are never $\mathbf{C}$-rational, and was later applied to show the irrationality of several other classes of complex threefolds (see for instance [6]). The work of Clemens and Griffiths was extended by Murre [71] to algebraically closed fields of any characteristic different from 2.

More recently, we implemented the arguments of Clemens and Griffiths over arbitrary perfect fields k [7]. By exploiting the fact that the intermediate Jacobian may be isomorphic to the Jacobian of a smooth projective curve over $\overline{k}$ while not being so over k, we produced new examples of varieties over k that are $\overline{k}$-rational but not k-rational.

Hassett and Tschinkel [46] subsequently noticed that over a non-closed field k, the intermediate Jacobian carries further obstructions to k-rationality: if X is a smooth projective k-rational threefold, then not only is its intermediate Jacobian isomorphic to the Jacobian $\mathbf{Pic}^0(C)$ of a smooth projective curve C over k, but in addition, assuming for simplicity that C is geometrically connected of genus ≥ 2, the $\mathbf{Pic}^0(C)$-torsors associated with $\mathrm{Aut}(\overline{k}/k)$-invariant algebraic equivalence classes of codimension 2 cycles on X are of the form $\mathbf{Pic}^\alpha(C)$ for some $\alpha \in \mathbf{Z}$. When X is a smooth three-dimensional complete intersection of two quadrics, they used these obstructions in combination with the natural identification of the variety of lines of X with a torsor under the intermediate Jacobian of X, and with the work of Wang [82], to prove Theorem A when $k = \mathbf{R}$ [46, Theorem 36] (and later [45] over subfields of $\mathbf{C}$).

The aim of the present article is to extend these arguments to arbitrary fields.

Applications to k-rationality criteria for other classes of $\overline{k}$-rational threefolds appear in the work of Kuznetsov and Prokhorov [61].

So far, we have been imprecise about what we call the intermediate Jacobian of a smooth projective threefold X over k.

If $k = \mathbf{C}$, one can use Griffiths' intermediate Jacobian J^3X constructed by transcendental means. This is the original path taken by Clemens and Griffiths [15]. The algebraic part of Griffiths' intermediate Jacobian has been shown to descend to subfields $k \subseteq \mathbf{C}$ by Achter, Casalaina-Martin and Vial [3, Theorem B]; the resulting k-structure on J^3X is the one used in [45].

Over algebraically closed fields k of arbitrary characteristic, a different construction of an intermediate Jacobian $\mathrm{Ab}^2(X)$, based on codimension 2 cycles, was provided by Murre [72, Theorem A p. 226] (see also [54]). This cycle-theoretic approach to intermediate Jacobians had already been applied by him to rationality problems (see [71]).

Over a perfect field k, the universal property satisfied by Murre's intermediate Jacobian $\mathrm{Ab}^2(X_{\overline{k}})$ induces a Galois descent datum on $\mathrm{Ab}^2(X_{\overline{k}})$, thus yielding a k-form $\mathrm{Ab}^2(X)$ of $\mathrm{Ab}^2(X_{\overline{k}})$ [3, Theorem 4.4]. It is this intermediate Jacobian $\mathrm{Ab}^2(X)$, which coincides with J^3X when $k \subseteq \mathbf{C}$, that we used in [7].

Over an imperfect field k, one runs into the difficulty that Murre's definition of $\mathrm{Ab}^2(X_{\overline{k}})$ does not give rise to a $\overline{k}/k$-descent datum on $\mathrm{Ab}^2(X_{\overline{k}})$. Achter, Casalaina-Martin and Vial still prove, in [4], the existence of an algebraic representative $\mathrm{Ab}^2(X)$ for algebraically trivial codimension 2 cycles on X (see §1.2 of *op. cit.* for the definition). However, when k is imperfect, it is not known whether $\mathrm{Ab}^2(X)_{\overline{k}}$ is isomorphic to $\mathrm{Ab}^2(X_{\overline{k}})$. For this reason, we do not know how to construct on $\mathrm{Ab}^2(X)$ the principal polarization that is so crucial to the Clemens-Griffiths method.

To overcome this difficulty and prove Theorem A in full generality, we provide, over an arbitrary field k, an entirely new construction of an intermediate Jacobian.

Our point of view is inspired by Grothendieck's definition of the Picard scheme (for which see [33], [12, Chapter 8], [56]). With any smooth projective $\overline{k}$-rational threefold X over k, we associate a functor $\mathrm{CH}^2_{X/k,\mathrm{fppf}} : (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Ab})$ endowed with a natural bijection $\mathrm{CH}^2(X_{\overline{k}}) \xrightarrow{\sim} \mathrm{CH}^2_{X/k,\mathrm{fppf}}(\overline{k})$ (see Definition 2.9 and (3.1)). The functor $\mathrm{CH}^2_{X/k,\mathrm{fppf}}$ is an analogue, for codimension 2 cycles, of the Picard functor $\mathrm{Pic}_{X/k,\mathrm{fppf}}$.

Too naive attempts to define the functor $\mathrm{CH}^2_{X/k,\mathrm{fppf}}$ on the category of k-schemes, such as the formula "$T \mapsto \mathrm{CH}^2(X_T)$", fail as Chow groups of possibly singular schemes are not even contravariant with respect to arbitrary morphisms: one would need to use a contravariant variant of Chow groups (see Remark 3.2 (ii)). To solve this issue, we view Chow groups of codimension ≤ 2 as subquotients of K-theory by means of the Chern character, and we define $\mathrm{CH}^2_{X/k,\mathrm{fppf}}$ as an appropriate subquotient of (the fppf sheafification of) the functor $T \mapsto \mathrm{K}_0(X_T)$. That this procedure gives rise to the correct functor, even integrally, is a consequence of the Riemann-Roch theorem without denominators [53].

We show that $\mathrm{CH}^2_{X/k,\mathrm{fppf}}$ is represented by a smooth k-group scheme $\mathbf{CH}^2_{X/k}$ (Theorem 3.1 (i)). Our functorial approach is crucial for this, as it allows us to argue by fppf descent from a possibly inseparable finite extension l of k such that X is l-rational. By construction, there is a natural isomorphism $\mathbf{CH}^2_{X_l/l} \simeq (\mathbf{CH}^2_{X/k})_l$ for all field extensions l of k.