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A DICHOTOMY FOR MINIMAL HYPERSURFACES IN MANIFOLDS THICK AT INFINITY

BY ANTOINE SONG

ABSTRACT. – Let (M, g) be a complete $(n+1)$-dimensional Riemannian manifold with $2 \leq n \leq 6$. Our main theorem generalizes the solution of S.-T. Yau's conjecture on the abundance of minimal surfaces and builds on a result of M. Gromov. Suppose that (M, g) has bounded geometry, or more generally is thick at infinity. Then the following dichotomy holds for the space of closed embedded hypersurfaces in (M, g): either there are infinitely many saddle points of the n-volume functional, or there is none.

Additionally, we give a new short proof of the existence of a finite volume minimal hypersurface in finite volume manifolds, we check Yau's conjecture for finite volume hyperbolic 3-manifolds and we extend the density result due to Irie-Marques-Neves when (M, g) is shrinking to zero at infinity.

RÉSUMÉ. – Soit (M, g) une variété riemannienne complète de dimension $n + 1$, où $2 \leq n \leq 6$. Notre théorème principal généralise la solution de la conjecture de S.-T. Yau concernant l'abondance des surfaces minimales, en s'appuyant sur un résultat de M. Gromov. Supposons que la géométrie de (M, g) est bornée, ou plus généralement que (M, g) est large à l'infini. Alors l'espace des hypersurfaces minimales compactes plongées de (M, g) satisfait la dichotomie suivante : ou bien il existe une infinité de points-selle du n-volume, ou bien il n'en existe aucun.

Par ailleurs, nous donnons une preuve nouvelle et courte du fait qu'une variété complète de volume fini contient une hypersurface minimale complète plongée de volume fini, nous vérifions la conjecture de Yau pour les 3-variétés hyperboliques de volume fini et nous étendons le résultat de densité de Irie-Marques-Neves au cas où (M, g) rétrécit vers zéro à l'infini.

1. Introduction

The search for minimal hypersurfaces in compact manifolds has enjoyed significant progress recently, thanks to the development of various min-max methods, such as the systematic extension of Almgren-Pitts' min-max theory [39] led by Marques and Neves

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[29, 31, 30, 32], the Allen-Cahn approach [18, 13, 5], or others [42, 25, 6, 52, 40, 38]. One central motivation was the following conjecture of S.-T. Yau:

YAU'S CONJECTURE ([50]). – *In any closed three-dimensional manifold, there are infinitely many minimal surfaces.*

Strong results implying the conjecture were obtained for generic metrics by Irie-Marques-Neves [24] (see [33] for a quantified version), Chodosh-Mantoulidis [5], X. Zhou [53], Y. Li [26]. Concurrently to these results, the conjecture for non-generic metrics was treated with a different line of arguments. When the manifold satisfies the "Frankel property," it was solved by Marques-Neves in [31]. We recently settled the general case in [45], where we localized min-max constructions appearing in [31] to some compact manifold with stable minimal boundary by introducing a non-compact manifold with cylindrical ends.

Yet, results about minimal hypersurfaces in complete non-compact manifolds are comparatively few and far between, and most of them are existence results. We give here a non-exhaustive list. In [9, 10], Collin-Hauswirth-Mazet-Rosenberg constructed a closed embedded minimal surface in any finite volume hyperbolic 3-manifold; there is also the work of Z. Huang and B. Wang [22], and of Coskunuzer [11]. In [35], Montezuma showed that a strictly mean concave compact domain in a complete manifold intersects a finite volume embedded minimal hypersurface. In [17], Gromov proved the following existence theorem, which we interpret as the analogue of Almgren-Pitts existence result [39] for non-compact manifolds:

GROMOV'S RESULT ([17]). – *In a complete non-compact manifold M, either there is an embedded finite volume complete minimal hypersurface, or there is a possibly singular strictly mean convex foliation of any compact domain of M.*

In [2], Chambers and Liokumovich showed the existence of a finite volume embedded minimal hypersurface in finite volume complete manifolds; in fact they proved the existence of such a minimal hypersurface if there is a region whose boundary is, say, ten times smaller than its width. In asymptotically flat 3-manifolds, Chodosh and Ketover constructed minimal planes in [4], using a degree argument (see Mazet-Rosenberg [34] for generalizations).

The goal of this paper is to propose a relevant generalization of the solution of Yau's conjecture to non-compact manifolds, by building on Gromov's result. Motivations came from our solution of the conjecture when the Frankel property is not satisfied [45], where we perform min-max in a non-compact manifold with cylindrical ends. Moreover some classes of manifolds naturally contain non-compact manifolds, for instance finite volume hyperbolic 3-manifolds. The non-compact situation substantially differs from the compact case: there are many non-compact manifolds without any closed (or finite volume) minimal hypersurfaces. For any integer $m > 0$, it is easy to construct a metric on $\mathbb{S}^2 \times \mathbb{R}$ for instance, with exactly m closed minimal surfaces. That metric can look like a long tube which gets thinner around $\mathbb{S}^2 \times \{0\}$, and the minimal surfaces are $\mathbb{S}^2 \times \{0\}$ and some other slices $\mathbb{S}^2 \times \{t\}$ which are degenerate stable. At first sight, it seems hard to come up with essentially different examples of manifolds that would contain only finitely many closed minimal hypersurfaces.

In our main result, we confirm this intuition for manifolds called "thick at infinity," which we define below.

The class $\mathcal{T}_\infty$ of manifolds thick at infinity

Before stating our main theorem, we recall the notion of "thickness at infinity" introduced by Gromov [17]. Minimal hypersurfaces in this paper are all *embedded* and unless mentioned, we consider hypersurfaces without boundary.

DEFINITION 1.1. – *Let (X^{n+1}, g) be a complete $(n+1)$-dimensional Riemannian manifold. (X, g) is said to be thick at infinity (in the weak sense) if any connected finite volume complete minimal hypersurface in (X, g) is closed. We denote by $\mathcal{T}_\infty$ the class of manifolds that are thick at infinity.*

In [17], Gromov actually uses a slightly stronger notion of thickness at infinity, since he asks that any connected finite volume minimal hypersurface with maybe non-empty compact boundary is compact.

The property of "thickness at infinity" is checkable. Complete manifolds with *bounded geometry* are important examples of manifolds thick at infinity. Some other examples are given in [17, Section 1.3], and the condition $\star_k$ in [35] also implies thickness at infinity. Easy special cases of the previous conditions include coverings of closed manifolds and asymptotically flat manifolds.

Note that M can be thick at infinity and at the same time "thin" in a certain sense. Indeed, using the monotonicity formula, it is easy to construct a warped product metric $g_t \oplus dt^2$ on a cylinder $N^n \times \mathbb{R}$ (where N is any closed n-dimensional manifold) such that $(N^n \times \mathbb{R}, g_t \oplus dt^2)$ is thick at infinity, has finite volume, and the n-volume of the cross section $N \times \{t\}$ decreases to zero as $t \to \pm\infty$.

A zero-infinity dichotomy for manifolds thick at infinity

Almost by definition, closed minimal hypersurfaces are critical points of the n-volume functional. By the properties of the Jacobi operator, which encodes the second variation of the n-volume at a minimal hypersurface, the space of deformations that do not increase the area at second order is finite dimensional. It is natural to define *saddle points of the n-volume functional* (or simply *saddle point minimal hypersurfaces*) as follows. Consider a connected closed embedded minimal hypersurface Γ. If it is 2-sided then we call it a saddle point minimal hypersurface if there is a smooth family of hypersurfaces $\{\Gamma_t\}_{t\in(-\varepsilon,\varepsilon)}$ $(\varepsilon > 0)$ which are small graphical perturbations of $\Gamma = \Gamma_0$ so that $\{\Gamma_t\}_{t\in(-\varepsilon,0)}$ and $\{\Gamma_t\}_{t\in(0,\varepsilon)}$ are on different sides of Γ and distinct from Γ, and

$$\mathrm{Vol}_n(\Gamma) = \max_{t\in(-\varepsilon,\varepsilon)} \mathrm{Vol}_n(\Gamma_t).$$

If Γ is 1-sided, we call it a saddle point minimal hypersurface if its connected double cover is a saddle point minimal hypersurface in a double cover of the ambient manifold. Note that if the metric is bumpy (i.e., no closed minimal hypersurface has a non-trivial Jacobi field), then saddle point minimal hypersurfaces are exactly unstable 2-sided closed minimal hypersurfaces and 1-sided closed minimal hypersurfaces with unstable double cover. By