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Kathryn MANN & Maxime WOLFF

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

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RECONSTRUCTING MAPS OUT OF GROUPS

BY Kathryn MANN AND Maxime WOLFF

Abstract. – We show that, in many situations, a homeomorphism f of a manifold M may be recovered from the (marked) isomorphism class of a finitely generated group of homeomorphisms containing f. As an application, we relate the notions of *critical regularity* and of *differentiable rigidity*, give examples of groups of diffeomorphisms of 1-manifolds with strong differential rigidity, and in so doing give an independent, short proof of a recent result of Kim and Koberda that there exist finitely generated groups of C^{α} diffeomorphisms of a 1-manifold M, not embeddable into $\mathrm{Diff}^{\beta}(M)$ for any $\beta > \alpha \geqslant 1$.

Résumé. – Nous montrons que dans de nombreuses situations, un homéomorphisme f d'une variété M peut être reconstruit à partir de la classe d'isomorphisme (marqué) d'un groupe de type fini d'homéomorphismes contenant f. Cela nous permet en particulier de relier la notion de *régularité critique* avec celle de *rigidité différentielle*, et nous donnons des exemples de groupes de difféomorphismes de variétés unidimensionnelles avec de fortes propriétés de rigidité différentielle. Nous en déduisons une preuve courte d'un résultat récent de Kim et Koberda, qui affirme qu'il existe des groupes de type fini d'homéomorphismes de classe C^{α} d'une variété unidimensionnelle M, qui n'admettent de plongement dans $\mathrm{Diff}^{\beta}(M)$ pour aucun $\beta > \alpha \geqslant 1$.

1. Introduction

1.1. Motivation

It is a classical and fundamental problem to describe to what extent the algebraic structure of a group determines the topological spaces on which the group can act, or constrains the regularity of those actions. For example, Whittaker [25] showed that connected, compact topological manifolds can be completely recovered from the algebraic structure of their groups of homeomorphisms: the existence of an isomorphism between $\mathrm{Homeo}(M)$ and $\mathrm{Homeo}(N)$ implies that $M = N$ and the isomorphism is an inner automorphism. This was

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generalized by Rubin to homeomorphism groups of other topological spaces, and Filipkiewicz [8] improved this to the groups of C^r diffeomorphisms of manifolds, showing that the algebraic structure of $\mathrm{Diff}^r(M)$ can even detect the regularity r.

All of these could be considered *recognition* or *reconstruction* theorems, showing that spaces can be recognized by their transformation groups. Another approach to this family of problems is to relate the complexity of a topological space to the algebraic complexity of (finitely generated) subgroups of its homeomorphism or diffeomorphism groups. This is, in some sense the "generalized Zimmer program," Zimmer's conjecture being that groups of high algebraic complexity, namely lattices of higher rank, cannot act by smooth or volume-preserving diffeomorphisms on low-dimensional manifolds.

This broad line of investigation has been particularly successful in dimension one. Here we know several purely algebraic conditions that prevent finitely generated groups from acting on one-manifolds with a given regularity. In the C^0 setting, this is the presence of left- or circular-orderability. In class C^1, many obstructions come from the *Thurston stability theorem*, while in higher regularity this program can be traced back all the way to Denjoy's work on rotations of the circle. To give some more recent examples, Navas [17] showed that Kazhdan's property T is an algebraic obstruction to acting on the circle with C^α regularity for $\alpha > 3/2$, and in [18] he showed that having intermediate growth is an obstruction to acting on the interval with regularity $\alpha > 1$. Castro-Jorquera-Navas [5] gave examples of nilpotent groups with sharp bounds on the Hölder regularity of their actions on the closed interval I; see also Jorquera-Navas-Rivas [13]. More recently, Kim and Koberda [14] gave examples of finitely generated subgroups of "critical regularity $\alpha \geqslant 1$," embeddable in $\mathrm{Diff}^\alpha(M)$ but not in $\mathrm{Diff}^\beta(M)$ for any $\beta > \alpha$ when $M = S^1$ or I.

1.2. Results

Our aim here is to contribute both to the general program of recognition and reconstruction, and to the problem of restricting regularity, with a specific application to the one-dimensional case.

We give general criteria for a group $\Gamma \subset \mathrm{Homeo}(X)$ of homeomorphisms of a space X to "reconstruct" or "recognize" other homeomorphisms of X purely through algebraic relations (Theorem 1.1). We also construct groups acting on 1-manifolds with a strong *differentiable rigidity* property (Theorem 1.5 and following), by using recent work of Bonatti-Monteverde-Navas-Rivas [1] and a precise version of the Sternberg linearization theorem. Building on all this, we deduce the existence of groups with *critical regularity* (Theorem 1.4). This gives an alternative short proof (and some generalization) of the critical regularity result of Kim and Koberda mentioned above. However, their techniques go further in a different direction than ours: they also give groups whose critical regularity passes to finite index subgroups, simple groups of given regularity, and define dynamical notions "δ-fast" and "λ-expansive" that are useful for explicitly constructing groups of specified regularity.

The remainder of this introductory section is devoted to giving precise statements of our results.

First result: map recognition

In general, if G is a group, $\Gamma \subset G$ is a subgroup and $g \in G$, we will say that Γ *recognizes* g, if for every element $h \in G$, the existence of a group isomorphism

$$\phi\colon \langle \Gamma, g\rangle \to \langle \Gamma, h\rangle$$

with $\phi|_\Gamma = \mathrm{id}_\Gamma$ and $\phi(g) = h$ implies that $h = g$. Essentially, Γ recognizes g if some equalities involving g and the elements of Γ characterize g among G. Note that if $\Gamma \subset \Gamma'$ then every element recognized by Γ is still recognized by Γ', and that every subgroup Γ recognizes at least its own elements. In this article we will consider the case when $G = \mathrm{Homeo}(X)$, where X is a topological space. If $\mathcal{C}$ is a subset of $\mathrm{Homeo}(X)$ we will say that Γ *recognizes maps in* $\mathcal{C}$, if it recognizes every element of $\mathcal{C}$. Typically, we will be interested in finitely generated groups Γ recognizing large classes of elements in $\mathrm{Homeo}(X)$.

The following theorem, proved in Section 2, shows that examples of such groups abound. We introduce some terminology needed for the statement. Recall that, for a group $\Gamma \subset \mathrm{Homeo}(X)$ and $\gamma \in \Gamma$, the *support* of γ is the closure of the set $\{x \in X \mid \gamma(x) \neq x\}$. *Non-total support* means $\mathrm{Supp}(\gamma) \neq X$. We say that Γ has *small supports everywhere* [(1)] if, for every nonempty open set $U \subset X$, there exists $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ with $\mathrm{Supp}(\gamma) \subset U$, and that Γ *has the contraction property* if, for any nonempty open set $U \subset X$, there exists $\gamma \in \Gamma$ such that $\gamma(X \setminus U) \subset U$.

THEOREM 1.1 (Map recognition). – *Let X be a Hausdorff topological space, and* $\Gamma \subset \mathrm{Homeo}(X)$.

(1) *If Γ has maps with* small supports everywhere*, then Γ recognizes all maps in* $\mathrm{Homeo}(X)$.

(2) *If Γ acts on X with the* contraction property*, then Γ recognizes homeomorphisms of X with non-total support.*

Similar conditions have been used elsewhere in the literature. The reconstruction theorems of Whittaker, Epstein, and Rubin [7, 25, 20] all use variations on the idea of small supports. To our knowledge, the contraction property was first used (under the more cumbersome name of "minimality and strong expansivity") in the proof by Margulis of the Tits' alternative in $\mathrm{Homeo}_+(S^1)$; see [15, 10].

We also show that Baumslag-Solitar groups give additional examples of groups with map recognition. These are needed for our applications and do not fall in the domain of Theorem 1.1.

THEOREM 1.2. – *The affine Baumslag-Solitar subgroup* $\mathrm{BS}(1,n) \subset \mathrm{Homeo}(\mathbb{R})$ *recognizes maps with compact support.*

Here $\mathrm{BS}(1,n)$ denotes the group generated by the maps $x \mapsto x+1$ and $x \mapsto nx$. Theorem 1.2 is proved in Section 3, where we actually prove something stronger—see Proposition 3.1. It would be interesting to find a simple and general condition that would simultaneously imply both the statements of Theorem 1.1 and Theorem 1.2.

(1) this is called *micro-supported* in [3, 4] and closely related to Rubin's notion of *locally moving* in [21]