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APPLICATIONS OF FORCING THEORY TO HOMEOMORPHISMS OF THE CLOSED ANNULUS

BY Jonathan CONEJEROS AND Fábio Armando TAL

ABSTRACT. – This paper studies homeomorphisms of the closed annulus that are isotopic to the identity from the viewpoint of rotation theory, using a newly developed forcing theory for surface homeomorphisms. Our first result is a solution to the so called strong form of Boyland's Conjecture on the closed annulus: Assume f is a homeomorphism of $\overline{\mathbb{A}} := (\mathbb{R}/\mathbb{Z}) \times [0,1]$ which is isotopic to the identity and preserves a Borel probability measure μ with full support. We prove that if the rotation set of f is a non-trivial segment, then the rotation number of the measure μ cannot be an endpoint of this segment. We also study the case of homeomorphisms such that $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times (0,1)$ is a region of instability of f. We show that, if the rotation numbers of the restriction of f to the boundary components lie in the interior of the rotation set of f, then f has uniformly bounded deviations from its rotation set. Finally, by combining this last result and recent work on realization of rotation vectors for annular continua, we obtain that if f is any area-preserving homeomorphism of $\overline{\mathbb{A}}$ isotopic to the identity, then for every real number ρ in the rotation set of f, there exists an associated Aubry-Mather set, that is, a compact f-invariant set such that every point in this set has a rotation number equal to ρ. This extends a result by P. Le Calvez previously known only for diffeomorphisms.

RÉSUMÉ. – Dans cet article, nous étudions les homéomorphismes de l'anneau compact qui sont isotopes à l'identité d'un point de vue de la théorie des rotations, en utilisant la notion de théorie de forçage récemment développée pour les homéomorphismes des surfaces. Notre premier résultat est une solution à la conjecture de Boyland sur l'anneau compact : Supposons que f est un homéomorphisme de $\overline{\mathbb{A}} := (\mathbb{R}/\mathbb{Z}) \times [0,1]$ qui est isotope à l'identité et qui préserve une mesure borélienne de probabilité μ à support total. Nous prouvons que si l'ensemble de rotation de f est un intervalle non trivial, le nombre de rotation de la mesure μ ne peut pas être une borne de cet intervalle. Nous étudions aussi les homéomorphismes f dont $\mathbb{A} := (\mathbb{R}/\mathbb{Z}) \times (0,1)$ est une région d'instabilité. Nous prouvons que si les nombres de rotation de la restriction de f aux composantes du bord appartiennent à l'intérieur de l'ensemble de rotation de f, alors la déviation de f de son ensemble de rotation est uniformément bornée. Enfin en combinant ce dernier résultat et des travaux récents de réalisation de vecteurs de rotation pour les anneaux continus, nous déduisons que si f est un homéomorphisme de $\overline{\mathbb{A}}$ qui est isotope à l'identité et qui préserve l'aire, alors pour tout nombre réel ρ dans l'ensemble de rotation de f il existe un ensemble d'Aubry-Mather, c'est-à-dire un ensemble compact et invariant tel que tout point dans cet ensemble a un nombre de rotation égal à ρ. Cela étend un résultat de P. Le Calvez connu auparavant uniquement pour les difféomorphismes.

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1. Introduction

This article studies homeomorphisms of the closed annulus that preserve the orientation and the boundary components, by the point of view of rotation theory. We denote by $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ the circle, by $\overline{\mathbb{A}} = \mathbb{T}^1 \times [0,1]$ the closed annulus and by $\widehat{\overline{\mathbb{A}}} = \mathbb{R} \times [0,1]$ its universal covering. Let $\widehat{\pi} : \widehat{\overline{\mathbb{A}}} \to \overline{\mathbb{A}}$ be the corresponding covering map, and $p_1 : \widehat{\overline{\mathbb{A}}} \to \mathbb{R}$ the projection on the first coordinate. Let $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a homeomorphism which preserves both orientation and boundary components and let $\widehat{f}$ be a lift of f to the universal covering. Inspired by the concept of Poincaré's rotation number for orientation-preserving homeomorphisms of the circle, one can define a similar object for $\widehat{f}$, called the *rotation set of* $\widehat{f}$, as follows: let μ be an f-invariant Borel probability measure on $\overline{\mathbb{A}}$. We can define the *rotation number of* μ *for* $\widehat{f}$ as

$$\operatorname{Rot}(\widehat{f}, \mu) := \int_{\overline{\mathbb{A}}} p_1(\widehat{f}(\widehat{z})) - p_1(\widehat{z})\, d\mu(z),$$

where $\widehat{z} \in \widehat{\pi}^{-1}(z)$. Note that this definition does not depend on the choice of $\widehat{z} \in \widehat{\pi}^{-1}(z)$. The *rotation set of* $\widehat{f}$, denoted by $\operatorname{Rot}(\widehat{f})$, is the set of all rotation numbers of f-invariant Borel probability measures. Since the set of f-invariant Borel probability measures is convex and compact in the weak-$*$topology, one shows that the rotation set of $\widehat{f}$ is a non-empty compact interval of $\mathbb{R}$.

We remark that the concept of rotation sets is not restricted to homeomorphisms of the annulus, and has been useful in the general study of homeomorphisms in the isotopy class of the identity of surfaces in general, and particularly for the two dimensional torus. One of the reasons for the growing interest in the subject is the variety of dynamical properties and phenomena that can be deduced from rotation sets; it is a useful tool in, for instance, estimating the topological entropy of a map in [35, 28] or determining the existence of periodic points with arbitrarily large prime periods and distinct rotational behavior in [11].

One of the driving problems in the understanding of the rotation theory for homeomorphisms of the closed annulus and of the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ has been the Boyland's Conjecture, see for instance [3, 38]. In the original form, Boyland's Conjecture for the closed annulus claimed that, whenever $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ preserved the Lebesgue measure and had a lift $\widehat{f}$ such that the rotation number of the Lebesgue measure for $\widehat{f}$ was null, then either the rotation set of $\widehat{f}$ was a singleton, or 0 lied in the interior of the rotation set of $\widehat{f}$. A stronger version of this conjecture has also been proposed, saying that whenever f preserved the Lebesgue measure and the rotation set of $\widehat{f}$ was a nondegenerate interval, then the rotation number of the Lebesgue measure for $\widehat{f}$ always lies in the interior of the rotation set, and similar questions were posed for homeomorphisms of $\mathbb{T}^2$. In [1] the strong form of Boyland's Conjecture for $\mathbb{T}^2$ was shown to hold for $\mathcal{C}^{1+\epsilon}$-diffeomorphisms, a result later extended for the $\mathcal{C}^0$ case in [32]. The later paper also proved the original conjecture for the closed annulus, but the strong version remained untenable. Our first result of this paper is the solution to this problem.

Theorem A. – *Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity and preserves a Borel probability measure μ with full support. Let $\widehat{f}$ be* *a lift of f to $\mathbb{R}\times[0,1]$. Suppose that* $\mathrm{Rot}(\widehat{f})$ *is a non-trivial segment. Then the rotation number of μ cannot be an endpoint of* $\mathrm{Rot}(\widehat{f})$.

Another research topic in rotation theory that has gathered substantial attention lately is the concept of bounded rotational deviations from rotation sets. It is a well known fact that, given an orientation-preserving homeomorphism $h:\mathbb{T}^1\to\mathbb{T}^1$ and a lift $\widehat{h}$ to the real line whose rotation number is α, one has that every orbit of $\widehat{h}$ remains at a bounded distance from the orbit of the associated rigid rotation. That is, there exists some constant $L>0$ such that, for all $\widehat{x}\in\mathbb{R}$ and all $n\in\mathbb{N}$, $|\widehat{h}^n(\widehat{x})-\widehat{x}-n\alpha|\le L$ (and in this case L can be taken as 1). A natural question is then to ask if some aspects of this property extend to similar situations for homeomorphisms of surfaces. For instance, one could pose the problem: consider a homeomorphism f of $\mathbb{T}^2$ in the isotopy class of the identity and say that f *has uniformly bounded deviations from its rotation set* if, given $\widehat{f}$ a lift of f to $\mathbb{R}^2$, the universal covering of $\mathbb{T}^2$, there is a constant $L>0$ such that, for all $\widehat{z}\in\mathbb{R}^2$ and all $n\in\mathbb{N}$, if d is the distance between a point and a set of $\mathbb{R}^2$, then $d(\widehat{f}^n(\widehat{z})-\widehat{z},n\,\mathrm{Rot}(\widehat{f}))\le L$. One then asks if it always holds that f has uniformly bounded deviations. This question is false in general, particularly when the rotation set of $\widehat{f}$ is a singleton (see for instance [21, 25]), but it does hold in many situations, particularly when $\mathrm{Rot}(\widehat{f})$ has nonempty interior (see [7, 8, 1, 14, 26, 32]), and similar results also are valid for homeomorphisms of $\mathbb{T}^2$ isotopic to Dehn Twists (see [2]). Furthermore, bounded deviations have also shown to have relevant dynamical consequences, for instance it was used in the proof of Boyland's Conjecture on $\mathbb{T}^2$ in [1, 32]. In some particular cases it can also imply that the dynamics factors over ergodic rotations of $\mathbb{T}^2$ (see [17]) or $\mathbb{T}^1$ (see [18] and [20]). Furthermore, it was shown in [39] that, for a class of $\mathcal{C}^r$ diffeomorphisms of $\mathbb{T}^2$, bounded deviations imply $\mathcal{C}^{r-1}$-rigidity, that is, that there exists a sequence of positive iterates of the map converging in the $\mathcal{C}^{r-1}$-topology to the identity.

Our second theorem deals with bounded deviations from rotation sets for homeomorphisms of $\overline{\mathbb{A}}$ in the following relevant scenario. We will say that $\mathbb{A}=\mathbb{T}^1\times(0,1)$ is a *Birkhoff region of instability* for a homeomorphism f of $\overline{\mathbb{A}}$ if for any neighborhood U of $\mathbb{T}^1\times\{0\}$ and any neighborhood V of $\mathbb{T}^1\times\{1\}$ one can find points $x\in U$, $y\in V$ and positive integers n_1,n_2 such that $f^{n_1}(x)\in V$ and $f^{n_2}(y)\in U$.

THEOREM B. – *Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}}=\mathbb{T}^1\times[0,1]$ which is isotopic to the identity. Suppose that $\mathbb{A}=\mathbb{T}^1\times(0,1)$ is a Birkhoff region of instability for f. Let $\widehat{f}$ be a lift of f to $\mathbb{R}\times[0,1]$. Suppose that $\mathrm{Rot}(\widehat{f})=[\alpha,\beta]$ and that both boundary component rotation numbers are strictly larger than α. Then there exists a real constant $L>0$ such that for every $\widehat{z}\in\mathbb{R}\times[0,1]$ and every integer $n\ge1$ we have*

$$p_1(\widehat{f}^n(\widehat{z}))-p_1(\widehat{z})-n\alpha\ge -L.$$

Likewise, if we assume that both boundary component rotation numbers are strictly smaller than β, then there exists a real constant $L>0$ such that for every $\widehat{z}\in\mathbb{R}\times[0,1]$ and every integer $n\ge1$ we have

$$p_1(\widehat{f}^n(\widehat{z}))-p_1(\widehat{z})-n\beta\le L.$$