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LOOP EQUATIONS AND A PROOF OF ZVONKINE'S qr-ELSV FORMULA

BY PETR DUNIN-BARKOWSKI, REINIER KRAMER, ALEXANDR POPOLITOV AND SERGEY SHADRIN

ABSTRACT. – We prove the 2006 Zvonkine conjecture that expresses Hurwitz numbers with completed cycles in terms of intersection numbers with the Chiodo classes via the so-called r-ELSV formula, as well as its orbifold generalization, the so-called qr-ELSV formula.

RÉSUMÉ. – Nous démontrons une conjecture de Zvonkine de 2006 qui exprime les nombres de Hurwitz avec cycles complétés en termes de nombres d'intersections avec des classes de Chiodo par la formule r-ELSV aussi bien que leur généralisation orbifolde, autrement dit la formule qr-ELSV.

1. Introduction

This paper is concerned with spin Hurwitz numbers, which have been conjectured by Zvonkine [51] to be expressable as integrals over the moduli space of curves, in a generalized ELSV formula, called Zvonkine's r-ELSV formula. In [39], the authors conjectured an orbifold generalization of this formula, called Zvonkine's qr-ELSV formula. In this paper, we prove the latter, and hence also the former, formula, via topological recursion and quadratic loop equations. We will introduce all of these concepts in this introduction.

1.1. q-orbifold r-spin Hurwitz numbers

In this section we introduce the q-orbifold r-spin Hurwitz numbers, following [45, 51, 47, 48, 39]. They are a very important and natural type of Hurwitz numbers; more precisely, they are a special case of completed Hurwitz numbers. Completed Hurwitz numbers were introduced by Okounkov and Pandharipande in [45] to establish a relation between Hurwitz numbers and relative Gromov-Witten invariants; in this section we recall their result specified for the q-orbifold r-spin case.

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1.1.1. *Completed cycles.* – A *partition* λ of an integer d is a non-increasing finite sequence $\lambda_1 \geq \cdots \geq \lambda_l$ such that $\sum \lambda_i = d$.

It is known that the irreducible representations ρ_λ of the symmetric group S_d are in a natural one-to-one correspondence with the partitions λ of d. On the other hand, to a partition λ of d one can assign a central element $C_{p,\lambda}$ of the group algebra $\mathbb{C}S_p$ for any positive integer p. The coefficient of a given permutation $\sigma \in S_p$ in $C_{p,\lambda}$ is defined as the number of ways to choose and label l cycles of σ so that their lengths are $\lambda_1, \ldots, \lambda_l$, and the remaining $p - d$ elements are fixed points of σ. Thus the coefficient of σ vanishes unless its cycle lengths are $\lambda_1, \ldots, \lambda_l, 1, \ldots, 1$. In particular, $C_{p,\lambda} = 0$ if $p < d$. Thus $C_{p,\lambda}$ is *the sum of permutations with l numbered cycles of lengths $\lambda_1, \ldots, \lambda_l$ and any number of non-numbered fixed points*.

The collection of elements $C_{p,\lambda}$ for $p = 1, 2, \ldots$ is called a *stable center element* C_λ. For example, the stable element $C_{(2)}$ is the sum of all transpositions in $\mathbb{C}S_p$, which is well-defined for each p, and in particular equals zero for $p = 1$.

Let λ be a partition of d and μ a partition of p. Consider the representation $\rho_\mu : S_p \to \mathrm{End}(V)$, V being the representation space with the dimension given by the hook length formula. Since $C_{p,\lambda}$ lies in the center of $\mathbb{C}S_p$, its image under ρ_μ (extended to the group algebra) is an operator, corresponding to a multiplication by a constant. Denote this constant by $f_\lambda(\mu)$. Thus to a stable center element C_λ we have assigned a function f_λ defined on the set of all partitions, $\mathcal{P}$. We are interested in the vector space spanned by the functions f_λ.

To study this space, one defines some new functions on the set of partitions as follows:

$$\mathbf{p}_{r+1}(\mu) = \frac{1}{r+1} \sum_{i \geq 1} \left[(\mu_i - i + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right] \qquad (r \geq 0). \tag{1.1}$$

(The standard definition [45, p.11] involves certain additive constants that we have dropped to simplify the expression, since these constants play no role in this paper.)

Theorem 1.1 (Kerov, Olshansky [37]). – *The vector space spanned by the functions f_λ coincides with the algebra generated by the functions $\mathbf{p}_1, \mathbf{p}_2, \ldots$*

As a corollary, to each stable center element C_λ we can assign a polynomial in $\mathbf{p}_1, \mathbf{p}_2, \ldots$ and, conversely, each $\mathbf{p}_{r+1}$ corresponds to a linear combination of stable center elements C_λ.

Definition 1.2. – The linear combination of stable center elements corresponding to $\mathbf{p}_{r+1}$ is called the *completed* $(r+1)$*-cycle* and denoted by $\overline{C}_{(r+1)}$.

The first completed cycles are:

$$\begin{aligned} \overline{C}_{(1)} &= C_{(1)}, \\ \overline{C}_{(2)} &= C_{(2)}, \\ \overline{C}_{(3)} &= C_{(3)} + C_{(1,1)} + \frac{1}{12} C_{(1)}, \\ \overline{C}_{(4)} &= C_{(4)} + 2C_{(2,1)} + \frac{5}{4} C_{(2)}, \\ \overline{C}_{(5)} &= C_{(5)} + 3C_{(3,1)} + 4C_{(2,2)} + \frac{11}{2} C_{(3)} + 4C_{(1,1,1)} + \frac{3}{2} C_{(1,1)} + \frac{1}{80} C_{(1)}. \end{aligned} \tag{1.2}$$

Definition 1.3. – We say that a stable center element C_λ involved in the completed cycle $\overline{C}_{(r+1)}$ has *genus defect* $[r+2-\sum(\lambda_i+1)]/2$.

1.1.2. *r-spin Hurwitz numbers.* – Let $g \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 1}$. Let $\vec{\mu} = (\mu_1, \dots, \mu_n)$ be an integer partition of length $n = \ell(\mu)$ such that $m := (\sum_{i=1}^n \mu_i + n + 2g - 2)/r$ is an integer, and let $d := |\mu| = \sum_{i=1}^n \mu_i$.

Recall that the completed $(r+1)$-cycle can be considered as a central element of the group algebra $\mathbb{C}S_d$. An *r-factorization of type* $(\mu_1, \dots, \mu_n)$ in the symmetric group S_d is a factorization

$$\sigma_1 \dots \sigma_m = \sigma \tag{1.3}$$

such that

(i) the cycle lengths of σ equal $\mu_1, \dots, \mu_n$ and

(ii) for each permutation σ_i, the stable center element corresponding to its cyclic type enters the completed $(r+1)$-cycle with a nonzero coefficient.

The product of these coefficients for i going from 1 to m is called the *weight* of the r-factorization.

Choose m points $y_1, \dots, y_m \in \mathbb{C}$ and a system of m loops $s_i \in \pi_1(\mathbb{C} \setminus \{y_1, \dots, y_m\})$, s_i going around y_i. Then to an r-factorization one can assign a family of stable maps from nodal curves to $\mathbb{C}P^1$. This is done in the following way.

(i) Consider the covering of $\mathbb{C}P^1$ ramified over $y_1, \dots, y_m$, and ∞ with monodromies given by $\sigma_1, \dots, \sigma_m$ and σ^{-1} (relative to the chosen loops).

(ii) If σ_i has l_i distinguished cycles and genus defect g_i, glue a curve of genus g_i with l_i marked points to the l_i preimages of the i-th ramification point that correspond to the distinguished cycles. The covering mapping is extended on this new component by saying that it is entirely projected to the i-th ramification point.

(iii) Among the newly added components, contract those that are unstable.

One can easily check that the arithmetic genus of the curve C constructed in this way is equal to g. The complex structure on the newly added components of C can be chosen arbitrarily, which implies that in general we obtain not a unique stable map, but a family of stable maps.

An r-factorization is called *transitive* if the curve C assigned to the factorization is connected. To give a more formal description, consider for each σ_k, $k = 1, \dots, m$, the set of transpositions T_{σ_k} that consists of $(ij) \in S_d$ for all i, j belonging to possibly different distinguished cycles of σ_k. Then an r-factorization is called *transitive* if the subgroup of S_d generated by σ and $\cup_{k=1}^m T_{\sigma_k}$ acts transitively on $\{1, \dots, d\}$.

Definition 1.4. – The *disconnected r-spin Hurwitz number* $h^{\bullet,r}_{g;\vec{\mu}}$ is the sum of weights of all r-factorizations of type $(\mu_1, \dots, \mu_n)$, divided by $|\mu|!m!$.

Definition 1.5. – The *connected r-spin Hurwitz number* $h^{\circ,r}_{g;\vec{\mu}}$ is the sum of weights of transitive r-factorizations of type $(\mu_1, \dots, \mu_n)$, divided by $|\mu|!m!$.

Note that connected and disconnected r-spin Hurwitz numbers are related via the usual inclusion-exclusion formula.