ISSN 0012-9593

quatrième série - tome 56 fascicule 4 juillet-août 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 56, 2023, p. 1231 à 1297

MOTIVIC COHOMOLOGY OF QUATERNIONIC SHIMURA VARIETIES AND LEVEL RAISING

BY RONG ZHOU

ABSTRACT. – We study the motivic cohomology of the special fiber of quaternionic Shimura varieties at a prime of good reduction. We exhibit classes in these motivic cohomology groups and use this to give an explicit geometric realization of level raising between Hilbert modular forms. The main ingredient for our construction is a form of Ihara's Lemma for compact quaternionic Shimura surfaces which we prove by generalizing a method of Diamond-Taylor. Along the way we also verify the Hecke orbit conjecture for these quaternionic Shimura varieties which is a key input for our proof of Ihara's Lemma.

RÉSUMÉ. – Nous étudions la cohomologie motivique de la fibre spéciale des variétés de Shimura quaternioniques à un nombre premier de bonne réduction. Nous explicitons des classes dans ces groupes de cohomologie motivique et utilisons cela pour donner une réalisation géométrique explicite de l'augmentation de niveau entre les formes modulaires de Hilbert. L'ingrédient principal de notre construction est une forme du lemme d'Ihara pour les surfaces de Shimura compactes quaternioniques, que nous prouvons en généralisant une méthode de Diamond-Taylor. En cours de route, nous vérifions également la conjecture de l'orbite de Hecke pour ces variétés de Shimura quaternioniques, qui est un point-clé de notre preuve du lemme d'Ihara.

1. Introduction

1.1. Main Theorem

The aim of this paper is to study the motivic cohomology of the special fiber of certain quaternionic Shimura varieties. For a scheme of finite type over a field, its motivic cohomology groups are a generalization of the usual Chow groups, and the main new observation of this paper is that for certain Shimura varieties, these groups can encode very rich arithmetic information. More precisely, we will show that the cycle class map from motivic cohomology to étale cohomology gives a geometric realization of level raising between Hilbert modular forms. The construction is also related to a geometric realization of the mod ℓ Jacquet-Langlands correspondence.

 doi:10.24033/asens.2554

We now state our main result. Let F be a totally real field of even degree $[F:\mathbb{Q}]=g$ and $p>2$ a prime which is *inert* in F. Let B be a totally indefinite quaternion algebra over F which is unramified at the unique prime $\mathfrak{p}$ above p and G the associated reductive group over $\mathbb{Q}$. Let K be a sufficiently small compact open subgroup of $G(\mathbb{A}_f)$ such that $K=K_pK^p$ where $K_p\subset G(\mathbb{Q}_p)=\mathrm{GL}_2(F_\mathfrak{p})$ is the standard hyperspecial maximal compact $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ and $K^p\subset G(\mathbb{A}_f^p)$. Then there is a Shimura variety $\mathrm{Sh}_K(G)$ defined over $\mathbb{Q}$; it extends to a smooth integral model $\underline{\mathrm{Sh}}_K(G)$ over $\mathbb{Z}_{(p)}$. We let $\mathscr{S}_K(G)$ denote its special fiber over $\mathbb{F}_p$ and $\mathscr{S}_K(G)_{\mathbb{F}_{p^g}}$ its base change to $\mathbb{F}_{p^g}$.

Fix an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_2(F)$ of parallel weight 2 defined over a number field $\mathbf{E}$. Let R be a finite set of places of F not containing $\mathfrak{p}$ and away from which Π is unramified and K is hyperspecial. We also choose a prime λ of $\mathcal{O}_\mathbf{E}$ whose residue characteristic is coprime to p and write $k_\lambda=\mathcal{O}_\mathbf{E}/\lambda$. We write $\mathrm{H}^i_\mathcal{M}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},k_\lambda(j))$ for the motivic cohomology group with k_λ coefficients defined in [40]. By [44], we may identify this with the higher Chow group $\mathrm{Ch}^j(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},2j-i,k_\lambda)$ defined in [2]. When $2j=i$, this group is just the usual Chow group of codimension j cycles modulo rational equivalence (with coefficients in k_λ).

The group $\mathrm{Ch}^j(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},2j-i,k_\lambda)$ is equipped with the following cycle class map to the absolute étale cohomology:

$$\mathrm{Ch}^j(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},2j-i,k_\lambda)\to \mathrm{H}^i_{\text{ét}}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},k_\lambda(j)). \tag{1.1.1}$$

We let $\mathbf{T}_\mathrm{R}$ denote the abstract Hecke algebra of $\mathrm{GL}_2(F)$ away from R; it is the $\mathbb{Z}$-algebra generated by elements $T_\mathfrak{q},S_\mathfrak{q}$ where $\mathfrak{q}$ runs over primes of F away from R. Then the Hecke eigenvalues of Π induce a map

$$\phi^\Pi_\lambda:\mathbf{T}_\mathrm{R}\to\mathcal{O}_\mathbf{E}\to k_\lambda.$$

We write $\mathfrak{m}_\mathrm{R}:=\ker(\phi^\Pi_\lambda)$ a maximal ideal of $\mathbf{T}_\mathrm{R}$ and we let $\mathfrak{m}\subset\mathbf{T}_{\mathrm{R}\cup\{\mathfrak{p}\}}$ be the preimage of $\mathfrak{m}_\mathrm{R}$ in $\mathbf{T}_{\mathrm{R}\cup\{\mathfrak{p}\}}$ under the natural inclusion $\mathbf{T}_{\mathrm{R}\cup\{\mathfrak{p}\}}\to\mathbf{T}_\mathrm{R}$.

The Hecke algebra $\mathbf{T}_{\mathrm{R}\cup\{\mathfrak{p}\}}$ acts on the étale cohomology $\mathrm{H}^\bullet_{\text{ét}}(\mathscr{S}_K(G)_{\overline{\mathbb{F}}_p},k_\lambda(-))$ and higher Chow groups $\mathrm{Ch}^j(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},2j-i,k_\lambda)$ of $\mathscr{S}_K(G)$. Upon making a large image assumption on the $\bmod\ \lambda$ Galois representation associated to Π (see Assumption 4.1.1) and localizing at the maximal ideal $\mathfrak{m}$, there is an isomorphism

$$\mathrm{H}^{g+1}_{\text{ét}}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},k_\lambda(g/2+1))_\mathfrak{m}\cong\mathrm{H}^1(\mathbb{F}_{p^g},\mathrm{H}^g_{\text{ét}}(\mathscr{S}_K(G)_{\overline{\mathbb{F}}_p},k_\lambda(g/2+1))_\mathfrak{m}).$$

The cycle class map then induces the *Abel-Jacobi map*:

$$\mathrm{Ch}^{g/2+1}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},1,k_\lambda)_\mathfrak{m}\to\mathrm{H}^1(\mathbb{F}_{p^g},\mathrm{H}^g_{\text{ét}}(\mathscr{S}_K(G)_{\overline{\mathbb{F}}_p},k_\lambda(g/2+1))_\mathfrak{m}). \tag{1.1.2}$$

In §5.3, we will define a subgroup $\mathrm{Ch}^{g/2+1}_{\mathrm{lr}}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},1,k_\lambda)_\mathfrak{m}$ of $\mathrm{Ch}^{g/2+1}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}},1,k_\lambda)_\mathfrak{m}$ using the geometry of Goren-Oort cycles (in fact the cycles we consider arise from the supersingular locus). on $\mathscr{S}_K(G)_{\mathbb{F}_{p^g}}$ as studied in [41], [42] and [27]. As the notation suggests, this subgroup is related to level raising. The main theorem of the paper is the following; we refer to §5.3 for the precise statement.

THEOREM 1.1.1. – *Suppose that p is a λ-level raising prime in the sense of Definition* 5.3.2 *and that Assumptions* 4.1.1 *and* 5.3.5 *are satisfied; in particular $T_{\mathfrak{p}} \equiv p^g + 1 \mod \mathfrak{m}_{\mathrm{R}}$ and $S_{\mathfrak{p}} \equiv 1 \mod \mathfrak{m}_{\mathrm{R}}$. Then the map*

$$\mathrm{Ch}^{g/2+1}_{\mathrm{lr}}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}}, 1, k_\lambda)/\mathfrak{m} \to \mathrm{H}^1(\mathbb{F}_{p^g}, \mathrm{H}^g(\mathscr{S}_K(G)_{\overline{\mathbb{F}}_p}, k_\lambda(g/2+1))/\mathfrak{m})$$

induced by (1.1.2) is surjective.

We note that as in [27, Remark 4.2, 4.6], if there exist rational primes inert in F, and Π is not dihedral and not isomorphic to a twist by a character of any of its internal conjugates, then for all but finitely many λ, the set of primes p which are λ-level raising primes has positive density. In general it is a difficult problem to produce non-zero classes in motivic cohomology. The key input to proving the surjectivity in Theorem 1.1.1 is a form of Ihara's Lemma which we prove by generalizing a method of Diamond-Taylor [8]; see the next subsection for more details.

We now give an example of the construction of $\mathrm{Ch}^{g/2+1}_{\mathrm{lr}}(\mathscr{S}_K(G)_{\mathbb{F}_{p^g}}, 1, k_\lambda)$ which makes clear the relationship with level raising. We assume $g = 2$ so that $\dim \mathscr{S}_K(G) = 2$.

We write B' for the totally definite quaternion algebra which agrees with B at all finite places. We fix an isomorphism

$$B' \otimes_{\mathbb{Q}} \mathbb{A}_f \cong B \otimes_{\mathbb{Q}} \mathbb{A}_f$$

which allows us to consider K as a compact open subgroup of $B' \otimes_{\mathbb{Q}} \mathbb{A}_f$. We let $\mathcal{X}'$ and $\mathcal{X}'_0(\mathfrak{p})$ denote the discrete Shimura sets where the compact open subgroup $K_0(\mathfrak{p}) \subset B' \otimes_{\mathbb{Q}} \mathbb{A}_f$ agrees with K away from $\mathfrak{p}$ and is the standard Iwahori subgroup of $\mathrm{GL}_2(F_{\mathfrak{p}})$ at $\mathfrak{p}$. We let

$$\pi_1, \pi_2 : \mathcal{X}'_0(\mathfrak{p}) \to \mathcal{X}'$$

denote the natural degeneracy maps so that the diagram

$$\mathcal{X}' \xleftarrow{\pi_1} \mathcal{X}'_0(\mathfrak{p}) \xrightarrow{\pi_2} \mathcal{X}'$$

is the usual Hecke correspondence for $\mathcal{X}'$. For any finite set S, we write $\Gamma(S, k_\lambda)$ for the abelian group of k_λ-valued functions on S.

We may think of $\mathscr{S}_K(G)$ as a moduli space of abelian varieties with multiplication by some maximal order $\mathcal{O}_B$ in B. We let $\mathscr{S}_K(G)^{\mathrm{ss}}$ be the locus where the underlying abelian variety is supersingular. Using the geometry of $\mathscr{S}_K(G)^{\mathrm{ss}}$ one can show that under the assumptions of Theorem 1.1.1, $\mathrm{Ch}^2(\mathscr{S}_K(G)_{\mathbb{F}_{p^2}}, 1, k_\lambda)_{\mathfrak{m}}$ admits a map from

$$\mathrm{K}_{\mathfrak{m}} := \ker\left((\pi_{1*}, \pi_{2*}) : \Gamma(\mathcal{X}'_0(\mathfrak{p}), k_\lambda) \to \Gamma(\mathcal{X}', k_\lambda)\right)_{\mathfrak{m}}.$$

The construction uses an interpretation of classes in $\mathrm{Ch}^2(\mathscr{S}_K(G)_{\mathbb{F}_{p^2}}, 1, k_\lambda)_{\mathfrak{m}}$ as codimension 1 cycles together with a rational function on the cycle; see §5.3 for the details. Then $\mathrm{Ch}^2_{\mathrm{lr}}(\mathscr{S}_K(G)_{\mathbb{F}_{p^2}}, 1, k_\lambda)_{\mathfrak{m}}$ is defined to be the image of $\mathrm{K}_{\mathfrak{m}}$. Theorem 1.1.1 in this case follows from the following stronger result:

THEOREM 1.1.2. – *Let $g = 2$. Suppose that p is a λ-level raising prime and that Assumption* 4.1.1 *is satisfied. Then the map*

$$\mathrm{K}_{\mathfrak{m}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^2_{\text{ét}}(\mathscr{S}_K(G)_{\overline{\mathbb{F}}_p}, k_\lambda(2))_{\mathfrak{m}}) \qquad (1.1.3)$$

is surjective.