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SEMICLASSICAL DIFFRACTION BY CONORMAL POTENTIAL SINGULARITIES

BY Oran GANNOT AND Jared WUNSCH

Abstract. – We establish propagation of singularities for the semiclassical Schrödinger equation, where the potential is conormal to a hypersurface. We show that a semiclassical wavefront set propagates along generalized broken bicharacteristics, hence reflection of singularities may occur along trajectories reaching the hypersurface transversely. The reflected wavefront set is weaker, however, by a power of h that depends on the regularity of the potential. We also show that for sufficiently regular potentials, the wavefront set does not stick to the hypersurface, but rather detaches from it at points of tangency to travel along ordinary bicharacteristics.

Résumé. – Nous établissons la propagation des singularités pour l'équation semi-classique de Schrödinger, où le potentiel est conormal à une hypersurface. Nous montrons que le front d'onde semi-classique se propage le long des bicaractéristiques généralisées brisées, d'où le reflet des singularités peut se produire sur les trajectoires atteignant transversement l'hypersurface. Le front d'onde réfléchi est pourtant plus faible, par une puissance de h qui dépend de la régularité du potentiel. Nous montrons également que pour des potentiels suffisamment réguliers, le front d'onde ne colle pas à l'hypersurface, mais doit se détacher aux points de tangence et continuer le long des bicaractéristiques ordinaires.

1. Introduction

1.1. Statement of results

Let (X, g) be a smooth n-dimensional Riemannian manifold, and $Y \subset X$ a hypersurface. We study propagation of semiclassical singularities for the Schrödinger operator

$$P = -h^2 \Delta_g + V, \tag{1.1}$$

where the real-valued potential V is conormal to Y. Semiclassical propagation of singularities theorems constrain the distribution of energy in phase space of a solution to (1.1), asymptotically as $h \to 0$: for V smooth, it is known that the energy concentrates on the classical energy surface and is invariant under the associated classical dynamics. Here, by contrast,

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the singularities of the potential V play an important role, diffracting energy along *broken* classical trajectories.

The class of potentials V that we consider are real-valued *conormal distributions* with respect to Y, a class of distributions that are smooth functions except at Y. If x is a defining function of Y then x_+^α is an instructive example, with $\alpha > 0$. More generally, we assume throughout that $V \in I^{[-1-\alpha]}(Y)$ for some $\alpha > 0$. This means that V is locally the inverse Fourier transform of a Kohn-Nirenberg symbol of order $-1-\alpha$, transverse to Y. In particular, V is $1+\alpha$ orders more regular than the delta distribution along Y. If $\alpha \geq k+\gamma$ with $k \in \mathbb{N}$ and $\gamma \in (0,1)$, then $V \in \mathcal{C}^{k,\gamma}(X)$, but V is $\mathcal{C}^\infty$ away from Y. (See Section 2.1 below for details.)

Let $p = |\xi|_g^2 + V$ denote the semiclassical principal symbol of P. Let H_p denote its associated Hamilton vector field, e.g., $\mathsf{H}_p = 2\xi \cdot \partial_x - (\partial_x V)\cdot \partial_\xi$ if g is the Euclidean metric. Recall that $\mathrm{WF}_h^s(u)$, the semiclassical wavefront set of order s, measures where, in T^*X, the family u fails to be $\mathcal{O}_{L^2}(h^s)$. If $Pu = 0$, then known results imply that the semiclassical wavefront set $\mathrm{WF}_h^s(u)$ of order s is contained in the characteristic set $\Sigma \equiv \{p = 0\}$, and is invariant under the H_p flow for each $s \in \mathbb{R} \cup \{+\infty\}$, at least away from Y. This result breaks down for singularities striking T_Y^*X: the conormal singularity of V causes ray splitting, generating wavefront set along both the reflected and transmitted components.

To make the notion of ray-splitting precise, we introduce a suitable *generalized broken bicharacteristic* (GBB) flow, taking into account both transverse and tangential incidence to Y. Properties of this GBB flow are described in detail in Section 4.3; its main feature is that the allowed trajectories are continuous in space but potentially discontinuous in momentum, with momentum tangent to Y conserved at interactions with this hypersurface, in accordance with the laws of reflection and refraction. The GBB flow is, consequently, not defined on the usual cotangent bundle, where it would be discontinuous. Instead, we introduce an adapted notion of semiclassical wavefront set by using a variant of Melrose's *b-calculus* of pseudodifferential operators. This gives rise to a *semiclassical b-wavefront set* which lives in a rescaling of the usual cotangent bundle, and agrees with the usual semiclassical wavefront set away from Y, but has the combined virtue and defect of not distinguishing different normal momenta over Y itself. The compressed characteristic set employed below is likewise an appropriately rescaled version of the set $\{p = 0\}$, which does not distinguish among different normal momenta over Y. (For details, including the relevant notation, see Section 3.)

THEOREM 1 (Propagation of singularities). – *Let* $\alpha > 0$ *and* $s \in \mathbb{R} \cup \{+\infty\}$. *If* u *is* h*-tempered in* $H_{h,\mathrm{loc}}^1(X)$*, then* $\mathrm{WF}_{\mathrm{b},h}^s(u) \setminus \mathrm{WF}_{\mathrm{b},h}^{-1,s+1}(Pu)$ *is the union of maximally extended* GBB*s within the compressed characteristic set* $\dot{\Sigma}$.

Suppose $Pu = 0$. Then Theorem 1 tells us that a given point in the wavefront set must give rise to wavefront set along *at least* one maximally extended GBB through it, but does not distinguish among the various possibilities. The theorems that follow draw subtler distinctions among them, and in particular give a special role to GBBs that are in fact ordinary solutions to Hamilton's equations of motion. Thus we now return to the usual cotangent bundle, where we may consider the usual Hamilton flow provided that there is enough regularity for it to make sense. Introduce local coordinates (x, y) such that $Y = \{x = 0\}$, and let (x, y, ξ, η) be the corresponding canonical coordinates on T^*X. Even though Hamilton's

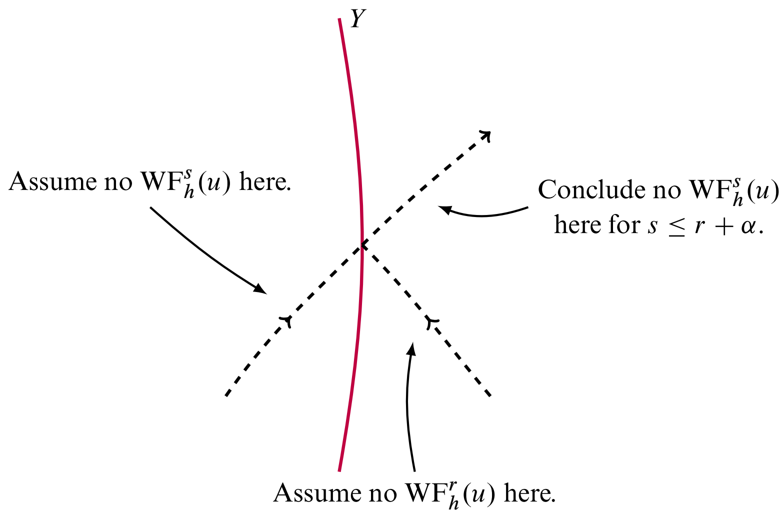

FIGURE 1. Illustration of the diffractive improvement. The trajectory at lower left is $\gamma_+((-\varepsilon, 0))$; its continuation across the interface is $\gamma_+((0, \varepsilon))$. The other incident trajectory at lower right is $\gamma_-((-\varepsilon, 0))$. The limitation on the propagation of regularity through the interface is $s \leq r + \alpha$.

equations become singular over Y when $\alpha \leq 1$, the integral curves of H_p are well defined near transversally incident points

$$\varpi_\pm = (0, y_0, \pm\xi_0, \eta_0) \in \Sigma,$$

where the normal momentum $\pm\xi_0$ does not vanish; see Lemma 4.2. The integral curves $\gamma_\pm$ with $\gamma_\pm(0) = \varpi_\pm$ therefore exist on some interval $(-\varepsilon, \varepsilon)$. To use the terminology of [18], the points $\varpi_\pm$ are said to be related, in the sense of having the same tangential momentum. Since $\mathrm{WF}_{\mathrm{b},h}^s(u) = \mathrm{WF}_h^s(u)$ away from Y, Theorem 1 states the following at transversally incident points: if $\gamma_+((-\varepsilon, 0))$ and $\gamma_-((-\varepsilon, 0))$ are both disjoint from $\mathrm{WF}_h^s(u)$, then

$$\gamma_+((0, \varepsilon)) \cap \mathrm{WF}_h^s(u) = \emptyset. \tag{1.2}$$

Nevertheless, the reflected singularity (namely the contribution of incident wavefront set along $\gamma_-((-\varepsilon, 0))$ to outgoing wavefront set along $\gamma_+((\varepsilon, 0)))$ is expected to be weaker than the original incident singularity along $\gamma_-((-\varepsilon, 0))$. In other words, if $\gamma_+((-\varepsilon, 0))$ is disjoint from $\mathrm{WF}_h^s(u)$ and $\gamma_-((-\varepsilon, 0))$ is disjoint from $\mathrm{WF}_h^r(u)$, then (1.2) should hold for a range of s depending on α and r. We show that at least when $\alpha > 1$, this holds for $s \leq r + \alpha$.

THEOREM 2 (Diffractive improvement at transverse reflection). – *Let $\alpha > 1$ and $s \leq r + \alpha$, where $s, r \in \mathbb{R} \cup \{+\infty\}$. Suppose that u is h-tempered in $H^1_{h,\mathrm{loc}}(X)$ with $Pu \in L^2_{\mathrm{loc}}(X)$, and $\mathrm{WF}_h^{s+1}(Pu) = \emptyset$. Let*

$$\varpi_\pm = (0, y_0, \pm\xi_0, \eta_0) \in \Sigma$$

with $\xi_0 \neq 0$, and let $\gamma_\pm$ be as above. If $\varpi_+ \in \mathrm{WF}_h^s(u)$, then there exists $\varepsilon > 0$ such that

$$\gamma_+((-\varepsilon, 0)) \subset \mathrm{WF}_h^s(u) \text{ or } \gamma_-((-\varepsilon, 0)) \subset \mathrm{WF}_h^r(u).$$