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ON BULK DEVIATIONS FOR THE LOCAL BEHAVIOR OF RANDOM INTERLACEMENTS

BY Alain-Sol SZNITMAN

ABSTRACT. – We investigate certain large deviation asymptotics concerning random interlacements in $\mathbb{Z}^d$, $d \geq 3$. We find the principal exponential rate of decay for the probability that the average value of some suitable non-decreasing local function of the field of occupation times, sampled at each point of a large box, exceeds its expected value. We express the exponential rate of decay in terms of a constrained minimum for the Dirichlet energy of functions on $\mathbb{R}^d$ that decay at infinity. An application concerns the excess presence of random interlacements in a large box. Our findings exhibit similarities to some of the results of van den Berg-Bolthausen-den Hollander in their work on moderate deviations of the volume of the Wiener sausage. An other application relates to recent work of the author on macroscopic holes in connected components of the vacant set.

RÉSUMÉ. – Nous étudions certaines asymptotiques de grandes déviations pour les entrelacs aléatoires sur $\mathbb{Z}^d$, $d \geq 3$. Nous déterminons le taux principal de décroissance exponentielle pour la probabilité que la valeur moyenne d'une fonction croissante au sens large du champ des temps d'occupation, échantillonnée en chaque point d'une grande boîte, dépasse son espérance. Nous exprimons le taux de décroissance exponentielle en termes d'un minimum sous contrainte de l'énergie de Dirichlet de fonctions sur $\mathbb{R}^d$ qui s'annulent à l'infini. Cela s'applique au cas d'une présence excessive des entrelacs dans une grande boîte. Nos résultats dans cet exemple présentent des similarités avec certains de ceux de van den Berg-Bolthausen-den Hollander dans leur article sur les déviations modérées du volume de la saucisse de Wiener. Une autre application a trait aux travaux récents de l'auteur concernant les trous macroscopiques dans les composantes de l'ensemble vacant.

0. Introduction

Random interlacements have deep links with random walks and with the Gaussian free field, see [28], [13], and in some respects behave as models of statistical mechanics having a continuous symmetry, such as the (massless) Gaussian free field. In the present work we investigate certain large deviation asymptotics related to the occupation times of the continuous-time random interlacements on $\mathbb{Z}^d$, $d \geq 3$. In essence, we aim at finding

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the principal exponential rate of decay of the probability that the average value of a non-decreasing local function of the field of occupation times of random interlacements at level $u > 0$, sampled at each point of a large box of $\mathbb{Z}^d$ centered at the origin, exceeds its expected value. For instance, if $I^u \subseteq \mathbb{Z}^d$ stands for the random interlacements at level u, we establish a formula for the principal exponential rate of decay of the probability that the fraction of sites of I^u in a large box centered at the origin exceeds a value ν that is bigger than $1 - e^{-u/g(0,0)}$ (with $g(\cdot,\cdot)$ the Green function of the simple random walk on $\mathbb{Z}^d$), i.e., bigger than the probability that the origin lies in I^u. As an other illustration, given any integer R, we establish a formula for the principal exponential rate of decay of the probability that in a large box centered at the origin, the fraction of sites x that are disconnected by I^u from the sphere $S(x,R)$ of sites at sup-distance R from x exceeds a value ν that is bigger than the probability $\mathbb{P}[0 \overset{V^u}{\not\leftrightarrow} S(0,R)]$ that 0 gets disconnected from $S(0,R)$ by I^u ($V^u = \mathbb{Z}^d \backslash I^u$ is the so-called vacant set of random interlacements). The principal exponential rates of decay that we obtain are expressed in terms of constrained minima for the Dirichlet energy of functions on $\mathbb{R}^d$ that decay at infinity. The first example mentioned above exhibits similarities with some of the results obtained by van den Berg-Bolthausen-den Hollander in their work [3] on moderate deviations of the volume of Wiener sausage. The second example is related to the recent work of the author concerning macroscopic holes in the connected components of the vacant set of random interlacements in the strongly percolative regime, see [32].

We now describe our results in more details. We consider $\mathbb{Z}^d, d \geq 3$, and, given $u \geq 0$, denote by $(L^u_x)_{x\in\mathbb{Z}^d}$ the field of occupation times of continuous-time random interlacements at level u, by I^u the random interlacements at level u, and by $V^u = \mathbb{Z}^d \backslash I^u$ the corresponding vacant set at level u (so that $I^u = \{x \in \mathbb{Z}^d; L^u_x > 0\}$ and $V^u = \{x \in \mathbb{Z}^d; L^u_x = 0\}$). We denote by $\mathbb{P}$ the probability measure governing these objects, and by $\mathbb{E}$ the corresponding expectation. We refer to [7], [14], Section 1 of [30], and below (1.11) for further details and references.

We probe the field of occupation times with the help of local functions. Each local function F comes with a non-negative integer R (its range), and is a function on $[0,\infty)^{B(0,R)}$ (with $B(0,R) = \{x \in \mathbb{Z}^d; |x|_\infty \leq R\}$ the closed ball in sup-distance with center 0 and radius R). Throughout this article, we assume that F satisfies (2.1). In essence, this condition requires that F is non-decreasing in each variable, $F(0) = 0$, F satisfies a sub-linear growth condition (which incidentally is automatically fulfilled when F is bounded), and it then follows from (2.1) (see below (2.1)) that the map

$$u \geq 0 \mapsto \theta(u) = \mathbb{E}\big[F\big((L^u_y)_{|y|_\infty \leq R}\big)\big] \geq 0 \text{ is continuous} \tag{0.1}$$

(θ is finite, non-decreasing, and vanishes for $u = 0$, as a result of the previously mentioned requirements on F). Some examples of local functions F of interest can be found in Example 2.1 of Section 2. For instance, $F_0(\ell) = \ell$ (with $R = 0$ and $\theta(u) = u$, see (1.12)) satisfies (2.1). More pertinent examples for our purpose are for instance

(0.2) $F(\ell) = 1\{\ell > 0\}$ (with $R = 0$ and $\theta(u) = 1 - e^{-u/g(0,0)}$, where $g(\cdot,\cdot)$ is the Green function of the simple random walk, see the beginning of Section 1).

and for $R \geq 0$,

$$
\text{(0.3)} \quad \begin{aligned} &F(\ell) = 1\{\text{any path in } B(0,R) \text{ from 0 to } S(0,R) \text{ meets a } y \text{ with } \ell_y > 0\} \\ &(\text{with } \ell \in [0,\infty)^R \text{ and } \theta(u) = \mathbb{P}[0 \overset{V^u}{\longleftrightarrow} S(0,R)] \text{ the probability that} \\ &I^u \text{ disconnects 0 from } S(0,R)). \end{aligned}
$$

Given $u > 0$ and a local function F as above (that is, satisfying (2.1)), one knows by the spatial ergodic theorem (see Theorem 2.8, p. 205 of [16], and also (4.4) of [22]) that

$$
\text{(0.4)} \quad \mathbb{P}\text{-a.s.}, \ \tfrac{1}{|B(0,N)|} \ \textstyle\sum_{x \in B(0,N)} F\big((L^u_{x+\cdot})\big) \underset{N \to \infty}{\longrightarrow} \theta(u) \quad (= \mathbb{E}[F((L^u_\cdot))]),
$$

where $|B(0,N)|$ stands for the number of sites in $B(0,N)$.

Actually, it is expedient to consider slightly more general sequences than $B(0,N)$, namely sequences obtained as the discrete blow-up of a model shape D in $\mathbb{R}^d$:

$$
\text{(0.5)} \qquad D_N = (ND) \cap \mathbb{Z}^d, \ N \geq 1,
$$

where we assume that

$$
\text{(0.6)} \quad \begin{aligned} &D \subseteq \mathbb{R}^d \text{ is the closure of a smooth bounded domain containing 0,} \\ &\text{or of an open } |\cdot|_\infty\text{-ball, which contains 0.} \end{aligned}
$$

We are interested by large deviation events of the *excess* type:

$$
\text{(0.7)} \qquad A_N = \Big\{ \sum_{x \in D_N} F\big((L^u_{x+\cdot})\big) > \nu\, |D_N| \Big\}, \ N \geq 1,
$$

where $\nu \in (0,\infty)$ is chosen bigger than $\theta(u)$. We have less to say on *deficit* deviation events, where the inequality in (0.7) is reversed and $\nu < \theta(u)$, see Remarks 4.6 and 5.5.

We derive asymptotic lower and upper bounds on $\frac{1}{N^{d-2}} \ \log \mathbb{P}[A_N]$ in our main Theorems 4.2 and 5.1. As an application of these theorems, we show in Corollary 5.9, that when F satisfying (2.1) is bounded, non-constant (this implies that θ is bounded and strictly increasing, see (2.2)), then, for $u > 0$, and $\theta(u) < \nu < \theta_\infty = \lim_{v \to \infty} \theta(v)$, one has

(0.8)

$$
\begin{aligned}
&\lim_N \tfrac{1}{N^{d-2}} \ \log \mathbb{P}[A_N] \\
&\quad = -\inf \Big\{ \tfrac{1}{2d} \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dz; \varphi \geq 0, \ \varphi \in C_0^\infty(\mathbb{R}^d) \text{ and } ⨍_D \theta\big((\sqrt{u} + \varphi)^2\big) \, dz > \nu \Big\} \\
&\quad = -\min \Big\{ \tfrac{1}{2d} \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dz; \varphi \geq 0, \ \varphi \in D^1(\mathbb{R}^d) \text{ and } ⨍_D \theta\big((\sqrt{u} + \varphi)^2\big) \, dz = \nu \Big\},
\end{aligned}
$$

where $⨍_D \, dz$ refers to the normalized integral $\frac{1}{|D|} \int_D \cdots dz$, with $|D|$ the Lebesgue measure of D, and $D^1(\mathbb{R}^d)$ stands for the space of locally integrable functions that vanish at infinity, with finite Dirichlet integral (see Chapter 8 §2 in [23]). One can actually omit the condition $\varphi \geq 0$ both in the infimum and the minimum in (0.8) (this is expedient when perturbing around a minimizer φ). Further, when $d \geq 5$, one can replace $D^1(\mathbb{R}^d)$ by the more traditional Sobolev space $H^1(\mathbb{R}^d)$, see Remark 5.10 1). Also, when D is a Euclidean ball, there is a spherically symmetric minimizer in the variational problem on the second line of (0.8), see Remark 5.10 2). Moreover, it is conceivable that the boundedness assumption made