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POSITROID VARIETIES AND CLUSTER ALGEBRAS

BY Pavel GALASHIN AND Thomas LAM

Abstract. – We show that the coordinate ring of an open positroid variety coincides with the cluster algebra associated to a Postnikov diagram. This confirms conjectures of Postnikov, Muller-Speyer, and Leclerc, and generalizes results of Scott and Serhiyenko-Sherman-Bennett-Williams.

Résumé. – On montre que l'anneau des fonctions régulières sur une variété positroïde coïncide avec l'algèbre amassée associée à un diagramme de Postnikov. Cela confirme des conjectures de Postnikov, Muller-Speyer, et Leclerc, et généralize des résultats de Scott et de Serhiyenko-Sherman-Bennett-Williams.

Positroid varieties are subvarieties of the Grassmannian that first appeared in the study of total positivity and Poisson geometry [16, 22, 3, 13]. In this paper we establish the following result; see Theorem 3.5.

Theorem. – *The coordinate ring* $\mathbb{C}[\mathring{\Pi}_{v,w}]$ *of an open positroid variety* $\mathring{\Pi}_{v,w}$ *is a cluster algebra.*

For the top-dimensional open positroid variety, this is due to Scott [24], a result that motivated much of the subsequent work. Combinatorially, positroid varieties are parametrized by Postnikov diagrams, and each such diagram gives rise to a quiver whose vertices are labeled by Plücker coordinates on the Grassmannian; see [22, 24]. This data gives rise to a cluster algebra of [5] whose cluster variables are rational functions on the Grassmannian, and since the work of Scott, it has been expected that this cluster algebra coincides with the coordinate ring of $\mathring{\Pi}_{v,w}$. This conjecture was made explicit by Muller and Speyer [21, Remark 4.6], and was established recently in the special case of Schubert varieties by Serhiyenko-Sherman-Bennett-Williams [25]. Another closely related conjecture was given

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by Leclerc [15], who constructed a cluster subalgebra of $\mathbb{C}[\mathring{\Pi}_{v,w}]$ using representations of preprojective algebras. We show (Corollary 3.8(i)) that these two cluster structures coincide. These cluster structures have also been compared in [25]; our work differs from theirs by switching from a left-sided to a right-sided quotient for the flag variety, i.e., from $B_-\backslash G$ to G/B_-; see Remark 3.2.

Leclerc's conjectures and results apply in the more general setting of open Richardson varieties. We hope to return to cluster structures of open Richardson varieties in the future. Some other closely related cluster structures include double Bruhat cells [2, 10], partial flag varieties [8], and unipotent groups [9].

Combining our main result with the well-developed machinery of cluster algebras has many consequences for the structure of open positroid varieties; see e.g., the introduction of [25]. For instance, the existence of a green-to-red sequence [6], together with the constructions of [11] endow $\mathbb{C}[\mathring{\Pi}_{v,w}]$ with a basis of *theta functions* with positive structure constants. Additionally, the results of [14] imply that $H^*(\mathring{\Pi}_{v,w}, \mathbb{C})$ satisfies the curious Lefschetz property, which has implications for extension groups of certain Verma modules that we aim to explore in future work.

Finally, we show that the totally nonnegative part $\Pi_{v,w}^{>0}$ of $\mathring{\Pi}_{v,w}$ (as defined by [16, 22]) is precisely the subset of $\mathring{\Pi}_{v,w}$ where all cluster variables take positive real values; see Corollary 4.4.

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Outline

We discuss the combinatorics of Le-diagrams in Section 1. The cluster algebra $\mathcal{A}(Q_D)$ coming from a Le-diagram D consists of some rational functions on the Grassmannian. As we discuss in Section 3.4, in order to prove our main result, one needs to show two inclusions: $\mathcal{A}(Q_D) \subseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$ and $\mathcal{A}(Q_D) \supseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$. For the first inclusion, we rely on the results of Leclerc [15]. In particular, following ideas of [25], we show in Section 2 that the cluster algebra of [15] is isomorphic to $\mathcal{A}(Q_D)$ (i.e., they have isomorphic quivers). We then prove the first inclusion $\mathcal{A}(Q_D) \subseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$ in Section 3; see Corollary 3.8(ii). We show the second inclusion $\mathcal{A}(Q_D) \supseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$ in Section 4, using the results of Muller-Speyer [21, 20], of Muller [19], and of Berenstein-Fomin-Zelevinsky [2].

Throughout the paper, we fix a positive integer n, and an integer $k \in [n] := \{1, 2, \dots, n\}$. For $a, b \in \mathbb{Z}$, we let $[a, b] := \{a, a+1, \dots, b\}$ if $a \leq b$, and $[a, b] := \emptyset$ otherwise.

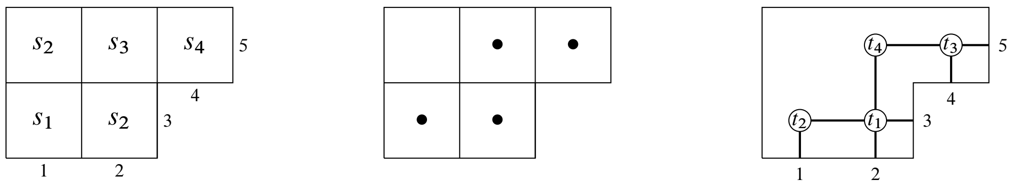

FIGURE 1. The Young diagram λ, Le-diagram D, and graph $G(D)$ corresponding to $(v, w) = (s_2, s_2 s_1 s_4 s_3 s_2)$

1. Le-diagram cluster algebra

Let $W = S_n$ be the symmetric group on n letters. For $i \in [n-1]$, let $s_i \in W$ denote the simple transposition of i and $i+1$. Every permutation $w \in W$ can be written as a reduced word $w = s_{i_1} \cdots s_{i_m}$ (where $m = \ell(w)$ is the length of w). In this case, $\mathbf{w} := (i_1, \dots, i_m)$ is called a *reduced expression* for w. We multiply permutations right-to-left; in particular, for $j \in [n]$ and $w = s_{i_1} \cdots s_{i_m}$, we let $w(j) := s_{i_1}(\dots(s_{i_m}(j))\dots)$. For $A \subset [n]$, we denote $wA := \{w(a) \mid a \in A\}$.

Let $J = [n] \setminus \{k\}$, and denote by $W^J \subset W$ the set of *k-Grassmannian permutations*, i.e., permutations $w \in W$ satisfying $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$. In other words, we have $w \in W^J$ if and only if $w = 1$ or each reduced word for w ends with s_k.

Let Q^J denote the set of pairs (v, w) where $w \in W^J$ and $v \leq w$ in the Bruhat order on W. The elements of Q^J label positroid varieties; see Section 3.1. By [17, Lemma 3.5], every reduced expression $\mathbf{w} = (i_1, \dots, i_m)$ for w contains a unique "rightmost" reduced subexpression $\mathbf{v}$ for v, called the *positive distinguished subexpression*. We let $J^\circ_{\mathbf{v}} \subset [m]$ denote the set of indices *not* used in $\mathbf{v}$.

1.1. Le-diagrams and subexpressions

We use English notation for Young diagrams and label their boxes in matrix notation. A *Le-diagram* D is a Young diagram λ, contained in a $k \times (n-k)$ rectangle, together with a filling of some of its boxes with dots, satisfying the following condition: if a box b is both below a dot and to the right of a dot, then b must contain a dot.

We describe a well-known bijection [22, Section 20] between elements of Q^J and Le-diagrams. First, Grassmannian permutations $w \in W^J$ are in bijection with Young diagrams $\lambda \subseteq k \times (n-k)$: placing s_{k+j-i} into the box (i, j) of λ, a reduced word for w is obtained by reading the boxes from right to left along each row, starting from the lowest row. The southeastern boundary edges of λ are labeled $1, 2, \dots, n$ from bottom-left to top-right. Thus the southern boundary edges are labeled by the elements of $w[k+1, n]$.

Given $v \leq w$, we mark the letters *not* used by the positive distinguished subexpression for v with a dot, and this gives a Le-diagram denoted $D(v, w)$. For example, if $(v, w) = (s_2, s_2 s_1 s_4 s_3 s_2)$, we have the Young diagram $\lambda = (3, 2)$ and the Le-diagram $D(v, w)$ in Figure 1(left and middle). Note that $w[k+1, n] = w\{3, 4, 5\} = \{1, 2, 4\}$ are the labels of the southern boundary edges.

Throughout the paper, we assume $(v, w) \in Q^J$ and denote $D := D(v, w)$. We also fix a choice of $\mathbf{w} = (i_1, \dots, i_m)$, $\mathbf{v}$, and $J^\circ_{\mathbf{v}}$ as above.