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LARGE DEVIATIONS FOR RANDOM WALKS ON GROMOV-HYPERBOLIC SPACES

BY Adrien BOULANGER, Pierre MATHIEU, Cagri SERT and Alessandro SISTO

Abstract. – Let Γ be a countable group acting on a geodesic Gromov-hyperbolic metric space X and μ a probability measure on Γ whose support generates a non-elementary subsemigroup. Under the assumption that μ has a finite exponential moment, we establish large deviations results for the distance and the translation length of a random walk with driving measure μ. From our results, we deduce a special case of a conjecture regarding large deviations of spectral radii of random matrix products.

Résumé. – Soient Γ un groupe dénombrable agissant sur un espace métrique géodesique hyperbolique X et μ une mesure de probabilité sur Γ dont le support engendre un semi-groupe non élémentaire. Sous l'hypothèse de moment exponentiel sur μ, on établit des résultats de grandes déviations pour le déplacement et la longueur de translation d'une marche aléatoire suivant la loi μ. Nous déduisons de nos résultats un cas particulier d'une conjecture concernant les grandes déviations des rayons spectraux de produits de matrices aléatoires.

1. Introduction

Let Γ be an infinite, countable group acting by isometries on a metric space (X, d), μ a probability measure on Γ and $z_0 \in X$ a base point. A (μ, z_0)-random walk on X, or random walk on X for short, is the image under the orbital map $\gamma \mapsto \gamma \cdot z_0$ of the random walk on Γ driven by the measure μ. We denote with $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma^{\mathbb{N}}$ (resp. $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$) the sequence of the successive positions of the walk on Γ (resp. the sequence of the successive positions of the image random walk on X). We refer to Section 3.1 for basics on random walks.

We will say that 'μ has a finite exponential moment' (resp. finite first moment), if the random variable $d(z_0, z_1)$ has a finite exponential moment (resp. finite first moment). In the sequel, $(\Omega, \mathbb{P})$ denotes the probability space on which the random walk is defined and $\mathbb{E}$ denotes the corresponding expectation.

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For a probability measure μ with finite first moment, the rate of escape of the random walk is defined as the limit

$$l := \lim_{n\to\infty} \frac{\mathbb{E}(d(z_n, z_0))}{n}. \tag{1.1}$$

(The existence of the limit follows from sub-additivity.) It follows from Kingman's sub-additive ergodic theorem that l is also the $\mathbb{P}$ almost sure limit of the ratio $d(z_n, z_0)/n$.

This article addresses the question of large deviations with respect to this last convergence: we are looking for estimates of the probability that the distance $d(z_n, z_0)/n$ deviates from l by an error of order 1, either from below or from above, and similarly for the translation length $\tau(\gamma_n)/n$ (see below for definitions). More precisely, we investigate the case where the space X is geodesic and Gromov-hyperbolic and the measure μ is *non-elementary*. A probability measure μ on Γ is said to be non-elementary when its support generates a subsemigroup which contains two independent loxodromic elements; see Subsection 3.3. Note that we do not assume that X is proper.

This setting has recently attracted a lot of attention as it encompasses several natural actions such as Gromov-hyperbolic groups acting on their Cayley graphs, rank-one semisimple groups acting on their symmetric spaces or Bruhat-Tits buildings, mapping class groups of surfaces acting on their curve complexes, relatively hyperbolic groups acting on their coned-off spaces, the Cremona group acting on the Picard-Manin hyperbolic space, etc. We refer to the introduction of [37, Section 1.2] for more details and references on the topic.

In [37], [46] and [38], the authors investigate the escape rate of random walks driven by non-elementary measures. They show in particular that it is positive in this setting. Their approach focuses on the boundary theory; they also manage to identify the Poisson boundary of the random walk with the Gromov boundary on the underlying Gromov-hyperbolic space under the assumption that the action is WPD. In [39] a different approach was proposed based on deviation inequalities (and thus without any reference to boundary theory). Under the assumption that the action is acylindrical, the authors manage to prove a central limit theorem for the rate of escape on the group itself.

1.1. Main results

To formulate our results on large deviations of random walks on X, recall that a sequence $(Z_n)_{n\in\mathbb{N}}$ of real-valued random variables is said to satisfy a *large deviation principle*, abbreviated LDP from now on, if there exists a lower-semicontinuous function, called the rate function, $I : \mathbb{R} \longrightarrow [0, \infty]$ such that for every measurable subset R of $\mathbb{R}$, we have

$$-\inf_{\alpha\in \mathrm{int}(R)} I(\alpha) \le \liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(Z_n \in R) \le \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(Z_n \in R) \le -\inf_{\alpha\in \overline{R}} I(\alpha), \tag{1.2}$$

where $\mathrm{int}(R)$ denotes the interior and $\overline{R}$ the closure of R. Our first main theorem is the following.

THEOREM 1.1. – *Let Γ be a countable group acting by isometries on a geodesic Gromov-hyperbolic space X, μ a non-elementary probability measure on Γ with finite exponential moment, and $z_0 \in X$. Then the sequence of random variables $(\frac{1}{n} d(z_0, z_n))_{n\in\mathbb{N}}$ satisfies a LDP with a proper convex rate function $I : [0, \infty) \to [0, \infty]$ which vanishes only at l.*

Note first that the rate function I does not depend on z_0 since the group acts by isometries. Indeed, for two different starting points z_0 and z'_0, the difference $|d(\gamma_n \cdot z'_0, z'_0) - d(\gamma_n \cdot z_0, z_0)|$ is bounded by $2\, d(z_0, z'_0)$. Below, we list some more remarks on this result:

REMARK 1.2. – 1. See Theorem 2.8 for a version of this result without any moment assumption on the probability measure μ and any hyperbolicity assumption on the metric space X.

2. By convexity and lower-semicontinuity of I, the effective support of I, namely the set $D_I = \{\alpha \in [0,\infty) \mid I(\alpha) < \infty\}$ is an interval and I is continuous on D_I. By Theorem 1.1, this in turn implies that for every subset J of D_I satisfying $\overline{\mathrm{int}(J)} = \overline{J}$ (e.g., any interval with non-empty interior), the limit $\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(\frac{1}{n} d(\gamma_n z_0, z_0) \in J)$ exists and is equal to $-\min_{\alpha\in\overline{J}} I(\alpha)$ (see Theorem 1.4 for more on D_I).

3. The assumption that μ has a finite exponential moment is sharp regarding the conclusion that the rate function I has unique zero (see Remark 3.2 and also Remark 2.9).

To the best of our knowledge, exponential decay in large deviations and LDP's had not been studied in the context of Theorem 1.1 so far. Even in the special case where Γ is Gromov-hyperbolic, Theorem 1.1 seems new. The most similar setting for which such a large deviation principle holds is for Lyapunov exponents associated to random products of matrices. We refer to the introduction of the third author's PhD thesis [43] and the references therein for more details. In particular, in that setting, the proof of exponential decay in large deviations (corresponding to uniqueness of the zero of I) goes back to Le Page [34].

When Γ is Gromov-hyperbolic and μ has a finite support, a possible alternative approach to prove that the rate function I has unique zero, would be to exploit the spectral gap property of the image of the random walk on the boundary of the group. We refer to [20, end of page 4]. For a surface group with the standard presentation and a driving measure with a finite exponential moment, large deviation estimates follow from the regeneration structure introduced in [23].

Another important geometric notion of size associated to an isometry γ acting on a Gromov-hyperbolic space (X,d) is its translation length defined as

$$\tau(\gamma) := \inf_{x\in X} d(x, \gamma\cdot x).$$

This quantity has the advantage not to depend on a base point and is a conjugacy invariant. Yet, it is perhaps harder to study than $d(x, g\cdot x)$ since it is not sub-additive. For example, the lack of sub-additivity prevents one from getting a convergence as in (1.1). On the other hand, it is known that for a non-elementary probability measure with bounded support, the averages $\frac{1}{n}\tau(\gamma_n)$ and $\frac{1}{n} d(z_n, z_0)$ behave similarly from the perspective of law of large numbers. Namely, they converge almost surely to the same constant l (see e.g., [38, Theorem 4.1]).

Let us now come to our second main theorem. We say that a set $\mathcal{B} \subset \mathrm{Isom}(X)$ is *bounded* if

$$\sup_{g\in\mathcal{B}} d(x, g\cdot x) < \infty$$

is bounded for some $x \in X$ (equivalently any). A probability measure μ on $\mathrm{Isom}(X)$ is said to have *bounded support* if its support is a bounded set. Our second main result reads