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BOUNDING THE BETTI NUMBERS OF REAL HYPERSURFACES NEAR THE TROPICAL LIMIT

BY Arthur RENAUDINEAU AND Kris SHAW

ABSTRACT. – We prove a bound conjectured by Itenberg on the Betti numbers of real algebraic hypersurfaces near non-singular tropical limits. These bounds are given in terms of the Hodge numbers of the complexification. To prove the conjecture we introduce a real variant of tropical homology and define a filtration on the corresponding chain complex inspired by Kalinin's filtration. The spectral sequence associated to this filtration converges to the homology groups of the real algebraic variety and we show that the terms of the first page are tropical homology groups with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The dimensions of these homology groups correspond to the Hodge numbers of complex projective hypersurfaces by combining results of Itenberg, Mikhalkin, Katzarkov, and Zharkov and the authors together with Arnal. The bounds on the Betti numbers of the real part follow, as well as a criterion to obtain a maximal variety. We also generalize Bertrand's formula relating the signature of the complex hypersurface and the Euler characteristic of the real algebraic hypersurface, as well as reprove Haas' combinatorial criterion for the maximality of plane curves near the tropical limit.

RÉSUMÉ. – On démontre une borne conjecturée par Itenberg sur les nombres de Betti des hypersurfaces algébriques réelles proches de la limite tropicale. Ces bornes sont exprimées en fonction des nombres de Hodge de la complexification. Pour démontrer ces bornes nous introduisons une variante de l'homologie tropicale dans le cadre réel, et définissons une filtration sur le complexe de chaîne associé, inspirée par la filtration de Kalinin. La suite spectrale associée à cette filtration converge vers les groupes d'homologie de la variété algébrique réelle, et nous montrons que les termes de la première page sont les groupes d'homologie tropicale (à coefficients dans $\mathbb{Z}/2\mathbb{Z}$). Les dimensions de ces groupes d'homologie correspondent aux nombres de Hodge des hypersurfaces projectives complexes grâce aux résultats d'Itenberg, Mikhalkin, Katzarkov, Zharkov d'une part et des auteurs avec Arnal d'autre part. Les bornes sur les nombres de Betti s'ensuivent, ainsi qu'un critère pour obtenir une variété maximale. Nous généralisons également la formule due à Bertrand reliant la signature d'une hypersurface complexe et la caractéristique d'Euler d'une hypersurface réelle, et nous redémontrons le critère combinatoire de Haas sur la maximalité des courbes planes proches de la limite tropicale.

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1. Introduction

A *real hypersurface* $V \subset \mathbb{P}^{n+1}$ of degree d is a hypersurface defined by a real homogeneous polynomial $f(z_0, \dots, z_{n+1}) \in \mathbb{R}[z_0, \dots, z_{n+1}]$ of degree d. We let $\mathbb{R}V$ denote the set of real points of V and $\mathbb{C}V$ denote the set of its complex points. The following fundamental question in real algebraic geometry can be traced back beyond Hilbert's sixteenth problem, see [23], [51], [16] for a survey. In this paper we denote by $\mathbb{Z}_2$ the field with two elements.

Question 1.1. – For any $0 \leq q \leq n$, what is the maximal possible value of the q-th Betti number

$$b_q(\mathbb{R}V) := \dim H_q(\mathbb{R}V; \mathbb{Z}_2)$$

among degree d non-singular real algebraic hypersurfaces V in $\mathbb{P}^{n+1}$?

In 1876, Harnack [22] proved the optimal bound

$$b_0(\mathbb{R}V) \leq \frac{(d-1)(d-2)}{2} + 1,$$

for V a non-singular real algebraic plane curve of degree d. This bound was generalized to any compact non-singular real curve by Klein [33] by replacing $\frac{(d-1)(d-2)}{2}$ with $g(\mathbb{C}V)$, where $g(\mathbb{C}V)$ denotes the genus of the complex curve. Beyond the case of plane curves, no optimal bounds are known in general on the individual Betti numbers of real algebraic varieties. For example, in the case of non-singular real algebraic surfaces in $\mathbb{P}^3$, the maximal values of the individual Betti numbers are unknown beyond degree 5. It is known that the maximal number of connected components of a non-singular real algebraic quintic surface is either 23, 24, or 25 and the maximal value of the first Betti number is either 45 or 47, see [38] and [32]. In general, we will denote by $h^{p,q}(\mathbb{C}V)$ the (p,q)-th Hodge number of $\mathbb{C}V$.

In relation to higher Betti numbers, in 1980 Viro formulated the following conjecture for all real projective surfaces.

Conjecture 1.2 (Viro). – *If V is a non-singular real projective surface such that $\mathbb{C}V$ is simply connected, then*

$$b_1(\mathbb{R}V) \leq h^{1,1}(\mathbb{C}V),$$

where $h^{1,1}(\mathbb{C}V)$ denotes the $(1,1)$-th Hodge number of $\mathbb{C}V$.

When V is the double covering of $\mathbb{P}^2$ ramified along a curve of even degree, the above conjecture is equivalent to the slight weakening of Ragsdale's conjecture about plane curves [42] made by Petrowsky [40], see also [48]. Counterexamples to Ragsdale's and Petrovsky's conjectures have been constructed by Viro [48] and Itenberg [25], respectively. Itenberg's counterexample paved the way to various counterexamples to Viro's conjecture and to constructions of real algebraic surfaces with many connected components, for example those in [26], [8], and [13]. It is still not known whether Viro's conjecture is true for surfaces which are maximal in the sense of the Smith-Thom inequality (1.2).

There are two main directions in Question 1.1. The first is to obtain restrictions on the topologies of real algebraic varieties, as is the case for Harnack's bound. The second direction is to provide constructions of real algebraic varieties with given topology. Viro's patchworking method provided a breakthrough in the second direction [49]. This technique continues to be the most powerful tool to construct real algebraic varieties in toric varieties with effectively computable topology. Here we will restrict our attention to Viro's primitive combinatorial patchworking, see Remark 3.8. The following was conjectured by Itenberg around 2005, and later appeared in [27].

CONJECTURE 1.3 ([27, Conjecture 2.5]). – *Let V be a real hypersurface in $\mathbb{P}^{n+1}$ obtained by a primitive patchworking. Then for any integer $q = 0, \dots, n$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) \text{ for } q = n/2, \\ h^{q,n-q}(\mathbb{C}V) + 1 \text{ otherwise.} \end{cases}$$

In the case of real algebraic surfaces in $\mathbb{P}^3$ arising from primitive patchworkings the above bounds were already proven by Itenberg [26], and are explicitly,

$$b_0(\mathbb{R}V) \leq \binom{d-1}{3} + 1 \qquad \text{and} \qquad b_1(\mathbb{R}V) \leq \frac{2d^3 - 6d^2 + 7d}{3}.$$

For example, real algebraic surfaces of degree 5 arising from a primitive patchworking satisfy $b_0(\mathbb{R}V) \leq 5$ and $b_1(\mathbb{R}V) \leq 45$. Furthermore, asymptotic analogues of the bounds in Conjecture 1.3 were proved by Itenberg and Viro in [29].

Viro's method for patchworking applies not only to real hypersurfaces in projective space but also to hypersurfaces in more general toric varieties, see for example [44]. Real algebraic hypersurfaces arising from primitive patchworking were later interpreted by Viro [50] as real algebraic hypersurfaces near non-singular tropical limits, see Definition 3.9 and [14, Section 5.3]. Here we will use this contemporary point of view on Viro's method and relate it to Viro's original formulation in Remark 3.8. A hypersurface near the non-singular tropical limit will always be (partially) compactified in a toric variety whose fan is a subfan of the dual fan of the Newton polytope of the hypersurface. We say a Newton polytope Δ is non-singular if the associated toric variety Y_Δ is non-singular. In this paper we establish the following theorem for real algebraic hypersurfaces in compact non-singular toric varieties near a non-singular tropical limit.

THEOREM 1.4. – *Let V be a compact real algebraic hypersurface with non-singular Newton polytope and near a non-singular tropical limit. Then for any integer $q = 0, \dots, n$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) \text{ for } q = n/2, \\ h^{q,n-q}(\mathbb{C}V) + h^{q,q}(\mathbb{C}V) \text{ otherwise.} \end{cases}$$

When the toric variety is a projective space, Conjecture 1.3 follows directly from the following theorem and the fact that $h^{q,q}(\mathbb{C}V) = 1$ for $2q \neq n$ by the Lefschetz hyperplane section theorem.

1.1. A guide to the proof of Theorem 1.4

To prove Theorem 1.4, we use a tropical description of primitive patchworking in terms of real phase structures, which we present in Section 3. We recall the relation to the standard version of primitive patchworking in Remark 3.8. For an n-dimensional non-singular real tropical hypersurface X in a tropical toric variety Y the notion of a real phase structure $\mathcal{E}$ is described in Definition 3.1. In Section 3.2 we describe a cellular cosheaf on X called the sign cosheaf $\mathcal{S}_\mathcal{E}$. A cellular cosheaf $\mathcal{G}$ on a tropical hypersurface X consists of a vector