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ON THE LONG TIME BEHAVIOR OF KDV TYPE EQUATIONS [after Martel-Merle]

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1. INTRODUCTION

A central problem in the theory of dispersive PDE's is to understand the interplay between nonlinearity and dispersion. In the context of the water waves problem (see *e.g.* [1]) the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + \partial_x(u^2) = 0, \quad x \in \mathbb{R} \tag{1}$$

appears to be the simplest (asymptotic) model where both dispersive and nonlinear effects are taken into account. If we neglect the nonlinear interaction $\partial_x(u^2)$ we deal with the Airy equation

$$u_t + u_{xxx} = 0. \tag{2}$$

The solutions of (2) are known to "disperse" in the sense that every solution u of (2) issued from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ initial data $u(0,\cdot)$, has its L^2 mass conserved but

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} = 0.$$

If we neglect the dispersive term u_{xxx}, we deal with the Burgers equation which is known to develop singularities in finite time, even for smooth initial data. The KdV equation (1) displays a balance between dispersion and nonlinearity since the dynamics of (1) is well defined, globally in time, for a very large class of initial data and moreover the solutions of (1) enjoy a rich dynamics as $t \to \infty$. A very special role among the solutions of (1) is played by the so-called solitary wave solution

$$u_c(t,x) = Q_c(x-ct) = \frac{3c}{2}\,\mathrm{ch}^{-2}\Big(\frac{\sqrt{c}}{2}(x-ct)\Big), \quad c > 0. \tag{3}$$

The solution (3) *does not disperse* and represents the displacement of the profile Q_c with speed c from left to right as the time t increases. Using the inverse scattering method (see [16, 29, 58]), it turns out that for sufficiently large t, any solution of (1) issued from well localized smooth initial data decomposes as a sum of solitary waves of type (3) plus a radiation term moving in the opposite direction. A natural generalization of (1), with stronger nonlinear effects, is the equation

$$u_t + u_{xxx} + \partial_x(u^p) = 0, \tag{4}$$

where p is a positive integer. The case $p = 3$ (modified KdV) is a very special case since, as in the case of (1), it can be treated with the inverse scattering method. Unfortunately, the integrability machinery does not seem to apply anymore for the equation (4) when $p \neq 2, 3$. Therefore the qualitative study of (4) in these cases is much less understood.

The equation (4) is a Hamiltonian PDE and its solutions enjoy, at least formally, the conservation laws

$$\|u(t,\cdot)\|_{L^2} = \|u(0,\cdot)\|_{L^2} \tag{5}$$

and

$$\frac{1}{2}\|u_x(t,\cdot)\|_{L^2}^2 - \frac{1}{p+1}\int_{-\infty}^{\infty} u^{p+1}(t,x)dx = \frac{1}{2}\|u_x(0,\cdot)\|_{L^2}^2 - \frac{1}{p+1}\int_{-\infty}^{\infty} u^{p+1}(0,x)dx. \tag{6}$$

Using the Gagliardo-Nirenberg inequalities

$$\|u(t,\cdot)\|_{L^{p+1}(\mathbb{R})}^{p+1} \leqslant C\|u(t,\cdot)\|_{L^2(\mathbb{R})}^{(p+3)/2}\|u_x(t,\cdot)\|_{L^2(\mathbb{R})}^{(p-1)/2},$$

we deduce from (5) and (6) that, for $p < 5$, the H^1 norm of $u(t,\cdot)$ is bounded independently of t as $u(0,\cdot) \in H^1(\mathbb{R})$. Consequently, the H^1 local well-posedness result of Kenig-Ponce-Vega [27] implies the existence of well-defined global dynamics of (4), for $p < 5$, in the energy space $H^1(\mathbb{R})$.

If $p \geqslant 5$, the H^1 local well-posedness result of Kenig-Ponce-Vega still applies (see Theorem 2.1 below) but the conservation laws (5), (6) provide no longer an H^1 control and hence solutions developing singularities in finite time may appear. The existence of such solutions has been a long standing open problem. In the case $p = 5$, this problem has been solved by Martel-Merle in a series of recent papers. The goal of this exposé is to discuss the main ideas developed by Martel-Merle, together with a presentation of previously known closely related results. One can extract from the results of Martel-Merle the following statement.

THEOREM 1.1 (Martel-Merle [35, 44, 36, 37]). — *Let $p = 5$. There exists $u_0 \in H^1(\mathbb{R})$ such that the local solution of* (4) *with initial data u_0 blows up in finite time. More precisely there exists $T > 0$ such that $\lim_{t\to T}\|u(t,\cdot)\|_{H^1(\mathbb{R})} = \infty$.*

We refer to section 8 below for a more precise statement. Let us make a comment on the choice of the initial data u_0. Equation (4) still has solutions of type (3). Namely, the solitary waves of (4) have the form $u_c(t,x) = Q_c(x - ct)$, $c > 0$ with $Q_c(x) = c^{1/(p-1)}Q(\sqrt{c}x)$ and

$$Q(x) = \Big[\frac{p+1}{2\,\mathrm{ch}^2\left(\frac{p-1}{2}x\right)}\Big]^{1/(p-1)}.$$

The crux of the Martel-Merle analysis is the deep understanding of the flow of (4) *close to a solitary wave.* It turns out that the solutions developing singularities in finite time constructed by Martel-Merle are issued from initial data close to $Q(x)$ and are essentially of the form $Q_{c(t)}(x + x(t))$ with $c(t) \to \infty$ as $t \to \infty$.

The study of solutions of PDE's developing singularities in finite time is an active research field. Let us briefly recall a few of the existing results and compare them with the analysis in the context of (4). In the case of semi-linear wave equations, due to the "finite propagation speed", the blow-up dynamics can be approximated by an ODE developing singularities in finite time (see [2] and the references therein). In the case of quasi-linear wave equations, a Burgers type behavior is behind the blow up dynamics (see [15] and the references therein). The equation (4) does not enjoy similar finite propagation speed properties and the qualitative study of (4) offers new features. Probably the closest models to (4) are the nonlinear Schrödinger equations (NLS). In the case of NLS, we have a functional (viriel functional) giving a simple obstruction for the existence of global dynamics (see [59] and the references therein). A similar functional is not known to exist in the context of (4). Due to a conformal invariance(1) of some Nonlinear Schrödinger equations, one can construct explicit blow-up solutions (see [41, 43, 60]). Similar invariance is not known in the context of (4).

The rest of this text is organized as follows. In the next section we recall some basic facts on the Cauchy problem for (4). Next, we recall results on the stability of the solitary waves for (4). Starting from section 4, we concentrate on the case $p = 5$. In section 4, we present a characterization of the solitary waves among the solutions with data close to the profile Q. Then, in sections 5 and 6, we present two applications of that characterization result. Section 5 is devoted to an asymptotic stability result while in section 6 we present a result showing the existence of solutions blowing up in finite or infinite time. The last two sections are devoted to the existence of solutions blowing up in finite time. In section 7, we present a result on the blow-up profile which is essential to prove the blow-up in finite time. Section 8 is devoted to the argument providing finite-time blow-up solutions. Finally, in section 9 we present some remarks and open problems.

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(1)The viriel functional is a consequence of that invariance too.

2. THE CAUCHY PROBLEM

In this section, we collect some preliminary results on the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \partial_x(u^p) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{7}$$

where $p \geqslant 1$ is an integer. The following theorem[(2)], which can be extracted from the work of Kenig-Ponce-Vega [27], is the starting point for the study of (7) in H^1.

THEOREM 2.1 ([27]). — *For every $u_0 \in H^1(\mathbb{R})$, there exist $T \in]0, +\infty]$, bounded from below by a positive constant which only depends on $\|u_0\|_{H^1}$, and a functional space X_T continuously embedded in $C([0,T]\,;H^1(\mathbb{R}))$ such that the Cauchy problem* (7) *has a unique maximal solution $u \in X_T$. Moreover, if $T < +\infty$ then $\lim_{t\to T} \|u(t,\cdot)\|_{H^1} = \infty$.*

Of course, a similar statement holds for negative times t. One can also prove the local well-posedness of (7) in H^s for suitable $s < 1$. This fact plays an important role in the Martel-Merle work. For example, it is used to prove that the flow enjoys a continuity property with respect to the weak H^1 topology.

Let us give some indications on the proof of Theorem 2.1 in the case $p = 5$. The proof of the other cases follows similar lines. In the case $p = 5$, one can prove that (7) is well-posed for data in H^s, $s > 0$. The proof is based on applying the contraction mapping principle to the integral formulation (Duhamel principle) of (7)

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)\partial_x(u^p(\tau))d\tau. \tag{8}$$

In (8), $S(t) = \exp(-t\partial_x^3)$ is the generator of the free evolution. This is the operator of convolution with respect to x with $(3t)^{-1/3}\,\mathrm{Ai}(x(3t)^{-1/3})$, where Ai is the Airy function. Let us recall that the Airy function is exponentially decaying on the right and it decays as $|x|^{-1/4}$ on the left (see *e.g.* [24]). Using the smoothing properties of $S(t)$ one can prove (see [27, Corollary 2.11]) that for $u_0 \in H^1$, the right-hand side of (8) is a contraction in a suitable ball of the space X_T of functions defined on $[0,T]\times\mathbb{R}$, equipped with the norm

$$\|u\|_{X_T} = \|u\|_{L^\infty_T H^s_x} + \|\mathrm{D}^s_x\, u\|_{L^5_x L^{10}_T} + \|\mathrm{D}^{s/3}_t\, u\|_{L^5_x L^{10}_T} + \|\mathrm{D}^s_x\, u_x\|_{L^\infty_x L^2_T} + \|\mathrm{D}^{s/3}_t\, u_x\|_{L^\infty_x L^2_T}.$$

The argument relies on some methods from harmonic analysis (restriction phenomena, maximal function estimates, etc.). In the case $s = 0$ the argument breaks down. However, in that case we are able to insure the contraction property, if $\|u_0\|_{L^2}$ is small enough. Therefore, if $p = 5$, the equation (4) is L^2-critical.

(2) We refer to [56, 7, 25, 21] for earlier results on the well-posedness theory of (7).

Another very important aspect in the study of (7) is the Kato smoothing effect (see [25]). Let $\varphi \in C^3(\mathbb{R})$ be bounded with all its derivatives. If u is a solution of (4) then, multiplying (4) with φu and integrating by parts, we obtain the formal[(3)] identity

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2(t)\varphi = -3\int_{-\infty}^{\infty} u_x^2(t)\varphi' + \int_{-\infty}^{\infty} u^2(t)\varphi^{(3)} + \frac{2p}{p+1}\int_{-\infty}^{\infty} u^{p+1}(t)\varphi'. \tag{9}$$

Note that, if φ is increasing then the first term in the right-hand side of (9) is negative. This fact was used by Kato [25] to show a remarkable local smoothing effect for (7), if $p < 5$. Namely the solution turns out to be one derivative smoother than the data, locally in space. In [25] well-posedness results in weighted Sobolev spaces are also obtained. The article of Kato was a great source of inspiration for many further works on the subject. It is also the case in the papers by Martel-Merle. For example, the crucial monotonicity properties (see section 5 below) are strongly related to identity (9).

3. STABILITY AND INSTABILITY OF THE SOLITARY WAVES

The initial data giving rise to blow-up solutions in the work of Martel-Merle belong to a small neighborhood of the function $Q(x)$ which is the initial data for a solitary wave. Thus the question of long time stability (or instability) of the solution $Q(x-t)$ of (4) is closely related to Martel-Merle analysis. This question has a long history starting from the pioneering work of Benjamin [4]. The aim of this section is to briefly summarize the state of the art on the stability of $Q(x-t)$. Similar discussion is valid for the solitary wave $Q_c(x-ct)$ (recall that $Q = Q_1$).

Let us first notice that there exist data for (4) arbitrary close to $Q(x)$ such that the corresponding solution does not stay close to $Q(x-t)$ for long times. This is clearly the case of $Q_c(x)$ with c close but different from 1. Indeed, if c is close to 1 then $Q(x)$ is close to $Q_c(x)$, but, because of the different propagation speed, $Q(x-t)$ and $Q_c(x-ct)$ separate from each other for $t \gg 1$.

Notice however that in the previous example the solution issued from Q_c remains close to spatial translates of Q. Hence this example does not exclude orbital stability of Q (up to the action of the group of spatial translations). Indeed, it turns out that for $p < 5$ the solution $Q(x-t)$ is orbitally stable under small H^1 perturbations. Here is the precise statement.

THEOREM 3.1. — *Let $p < 5$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if the initial data of* (7) *satisfies $\|u_0 - Q\|_{H^1} < \delta$ then there exists a C^1 function $x(t)$ such*

[(3)] The rigorous justification for sufficiently "nice solutions" u can be obtained by approximation arguments thanks to a propagation of regularity property of the local flow of (7).