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TYPE THEORY AND COEFFICIENT SYSTEMS ON THE BUILDING

by Paul Broussous & Peter Schneider

Abstract. — Let F be a non-archimedean local field and G be the group $\mathrm{GL}(N,F)$ for some integer $N \geq 2$. Let π be a smooth complex representation of G lying in the Bernstein block $\mathcal{B}(\pi)$ of some simple type in the sense of Bushnell and Kutzko [10]. Refining the approach of the second author and U. Stuhler in [18], we canonically attach to π a subset X_π of the Bruhat-Tits building X of G, as well as a G-equivariant coefficient system $\mathcal{C}[\pi]$ on X_π. Roughly speaking the coefficient system is obtained by taking isotypic components of π according to some representations constructed from the Bushnell and Kutzko type of π. We conjecture that when π has central character, the augmented chain complex associate to $\mathcal{C}[\pi]$ is a projective resolution of π in the category $\mathcal{B}(\pi)$. Moreover we reduce this conjecture to a technical lemma of representation theoretic nature. We prove this lemma when π is an irreducible discrete series of G. Following closely [19], we then attach to any irreducible discrete series π of G an explicit pseudo-coefficient f_π and obtain a Lefschetz type formula for the value of the Harish-Chandra character of π at a regular elliptic element. In contrast to that of [19], this formula allows explicit character value computations.

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RÉSUMÉ (*Théorie des types et système de coefficients sur l'immeuble*). — Soient F un corps local non archimédien et G le groupe $\mathrm{GL}(N,F)$, pour un entier $N \geq 2$. Soit π une représentation lisse complexe de G appartenant au bloc de Bernstein $\mathcal{B}(\pi)$ d'un type simple au sens de Bushnell et Kutzko [10]. En affinant l'approche que proposent le second auteur et U. Stuhler dans [18], nous attachons canoniquement à π un sous-ensemble X_π de l'immeuble de Bruhat-Tits X de G, ainsi qu'un système de coefficients G-équivariant $\mathcal{C}[\pi]$ sur X_π. Grossièrement parlant, le système de coefficients est construit en prenant des composantes isotypiques de π selon des représentations construites à partir du type de Bushnell et Kutzko de π. Nous conjecturons que lorsque π possède un caractère central, le complexe de chaînes augmenté associé à $\mathcal{C}[\pi]$ est une résolution de π dans la catégorie $\mathcal{B}(\pi)$. De plus nous réduisons cette conjecture à un lemme technique en théorie des représentations. Nous démontrons ce lemme lorsque π est une représentation irréductible de la série discrète de G. Ensuite, suivant de près [19], nous attachons à toute représentation irréductible π de la série discrète de G un pseudo-coefficient explicite f_π et obtenons une formule de type Lefschetz pour la valeur du caractère de Harish-Chandra de π en un élément elliptique régulier. Contrairement à celle obtenue dans [19], notre formule permet des calculs explicites.

Introduction

Let F be a non-archimedean local field, and for some integer $N \geq 2$, let G denote the locally compact group $\mathrm{GL}(N,F)$ and X its Bruhat-Tits building. The aim of this work is to refine the construction of [18] (also see [19]) to attach to certain representations of G new equivariant coefficient systems on the Bruhat-Tits building. These representations belong to the Bernstein blocks of the category of smooth complex representations of G corresponding to *simple types* in the sense of Bushnell and Kutzko [10]. Let $(\pi,\mathcal{V})$ be a smooth complex representation of G. In [18] an equivariant coefficient system $\mathcal{C}[\pi]$ is constructed by attaching to each simplex σ of X the space of vectors fixed by a certain congruence subgroup of level e of the parahoric subgroup of G fixing σ. Here the integer e is such that $\mathcal{V}$ is generated as a G-module by its vectors fixed by the principal congruence subgroup of level e of some maximal compact subgroup of G. In [18] it is proved that the augmented chain complex $C_\bullet(X,\mathcal{C}[\pi]) \longrightarrow \mathcal{V}$ of X with coefficients in $\mathcal{C}[\pi]$ is exact. If one moreover assumes that $(\pi,\mathcal{V})$ admits a central character χ, then $C_\bullet(X,\mathcal{C}[\pi]) \longrightarrow \mathcal{V}$ is a projective resolution of $(\pi,\mathcal{V})$ in the category of smooth representations of G with central character χ. In [3], the first author gave another proof of this fact for Iwahori-spherical representations. In [19], the second author and U. Stuhler draw some important consequences concerning the harmonic analysis on G as well as the homological algebra of the category of smooth representations of G. Among other things they prove that these projective resolutions give rise to pseudo-coefficients for discrete series representations (generalizing the pseudo-coefficient constructed by Kottwitz in [15] for the Steinberg representation) as well as a Lefschetz type character formula for the Harish-Chandra character of any smoooth representation. Note that if the construction of [18] is restricted to the group G, [19] gives a generalization to any connected reductive F-group $\mathbb{G}$ and most of its results are valid without restriction on G (but sometimes F is assumed to have characteritic 0, and $\mathbb{G}(F)$ to have compact center).

If the construction and results of [18], [19] have important theoretic consequences, they do not allow explicit calculations. Indeed in general the coefficient system $\mathcal{C}(\pi)$ cannot be explicitely computed (except may be in the *level* 0 *case*, but this is nowhere written). Indeed the only explicit way to be given an irreducible smooth representation of G is to specify its Bushnell and Kutzko type. This is why it is natural to seek for a refinement of [18] based on Bushnell and Kutzko theory.

In this paper, for technical reasons, we restrict to representations belonging to Bernstein blocks of G attached to simple types. These Bernstein blocks are exactly those containing discrete series representations. We fix a simple type (J, λ) and denote by $\mathcal{R}_{(J,\lambda)}(G)$ the category of smooth representations of G that are generated by their λ-isotypic component. We fix a smooth representation $(\pi, \mathcal{V})$ of G lying in $\mathcal{R}_{(J,\lambda)}(G)$. To the datum (J, λ), in a non canonical way, one may associate a field extension E/F of degree dividing N whose multiplicative group $E^\times$ is embedded in G. The centralizer G_E of $E^\times$ in G is isomorphic to $\mathrm{GL}(N/[E:F], E)$. Using a result of the first author and B. Lemaire [5], we may view the Bruhat-Tits building X_E of G_E as being embedded in X in a G_E-equivariant way. We show that *hidden* in the properties of *Heisenberg representations* constructed in [10]§ (5.1) and in the *mobility* of simple characters established in *loc. cit.* § (3.6), there is a *geometric structure* allowing to attach to π a G_E-equivariant coefficient system $\mathcal{C}_E[\pi]$ on the first barycentric subdivision $\mathrm{sd}(X_E)$ of X_E. More precisely, in a non canonical way, we attach to (J, λ) a collection of pairs $(J^1(\sigma,\tau), \eta(\sigma,\tau))_{\sigma\subset\tau}$, where σ and τ run over the simplices of X_E satisfying $\sigma \subset \tau$. Here $J^1(\sigma,\tau)$ is some compact open subgroup of G and $\eta(\sigma,\tau)$ a Heisenberg representation of $J^1(\sigma,\tau)$ as considered in *loc. cit.* (5.1.14) (but Bushnell and Kutzko do not use this language nor this notation). Moreover the collection $(J^1(\sigma,\tau), \eta(\sigma,\tau))_{\sigma\subset\tau}$ is G_E-equivariant. Exploiting the compatibility relations among the various $\eta(\sigma,\tau)$ proved in *loc. cit.* § (5.1), and by taking isotypic components of $\mathcal{V}$ according to the Heisenberg representations $\eta(\sigma,\tau)$, we construct our equivariant coefficient system $\mathcal{C}_E[\pi]$.

We then show that the subset $X[E]$ of X obtained by taking the union of the $g.X_E$, where g runs over G, has the structure of a G-simplicial complex containing X_E as a subcomplex. We naturally attach to $\mathcal{C}_E[\pi]$ a G-equivariant coefficient system $\mathcal{C}[\pi]$ on the first barycentric subdivision of $X[E]$ and show that it actually derives from a coefficient complex on $X[E]$, still denoted by $\mathcal{C}[\pi]$. We prove that the simplicial complex $X[E]$ and the coefficient system $\mathcal{C}[\pi]$ are actually independent of any choice made in their construction: these are