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p-ADIC AUTOMORPHIC FORMS ON REDUCTIVE GROUPS

by

Haruzo Hida

Abstract. — In these lecture notes, we will prove vertical control theorems for ordinary p-adic automorphic forms and irreducibility of the Igusa tower over unitary and symplectic Shimura varieties.

***Résumé* (Formes automorphes p-adiques sur les groupes réductifs).** — Nous démontrons le contrôle vertical pour les formes automorphes ordinaires p-adiques et l'irreductibilité de la tour d'Igusa pour les variétés de Shimura symplectique et unitaire.

Contents

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1. Introduction

Let p be a prime. What I would like to present in this series of lectures is the theory of families of p-ordinary p-adic (cohomological) automorphic forms on reductive groups. After going through basics of the theory of p-adic automorphic forms, we would like to study

(1) Vertical Control Theorem (VCT: construction of p-adic families);
(2) p-adic L-functions (in Symplectic and Unitary cases);
(3) Galois representations;
(4) the Iwasawa theoretic significance of p-adic L-functions.

1.1. Automorphic forms on classical groups. — Let $G_{/\mathbb{Z}}$ be an affine group scheme whose fiber over $\mathbb{Z}_p$ is a classical Chevalley group; so, unitary groups are included (dependent on the choice of p). Take a Borel subgroup B and its torus T. When G is split over $\mathbb{Q}$, we may embed G into $GL(n)_{/\mathbb{Q}}$. Let B be the Borel subgroup (we can take it to be the group of upper triangular matrices in G). Let T be the group of diagonal matrices. We have a splitting $B = T \ltimes U$ for the unipotent radical U of B. On the quotient variety G/U (which is a T-torsor over the projective flag variety G/B), T acts by $gUt = gtU$, and hence T acts on the structure sheaf $\mathcal{O}_{G/U}$ by $t\phi(gU) = \phi(gtU)$. This action gives rise to an order on $X(T) = \mathrm{Hom}(T_{/\overline{\mathbb{Q}}}, \mathbb{G}_m)$ so that the positive cone in $X(T)$ is made of $\kappa \in X(T)$ such that the κ-eigenspace $L(\kappa)$ on the global sections of $\mathcal{O}_{G/U}$ is non-trivial. We then have a representation $L(\kappa; A) = L_G(\kappa; A)$ on $L(\kappa)$ given by $\phi(gU) \mapsto \phi(h^{-1}gU)$ for $h \in G(A)$, as long as T is split over a ring A. When $G = SL(2)$, $T \cong \mathbb{G}_m$, $X(T) \cong \mathbb{Z}$ by $\kappa \leftrightarrow n$ if $\kappa(x) = x^n$, and $L(\kappa; A)$ is the symmetric κ-th tensor representation of $SL(2)$, which can be realized on the space of homogeneous polynomials of degree n so that $\alpha \in SL(2)$ acts on a polynomial $P(X,Y)$ by $P(X,Y) \mapsto P((X,Y)^t\alpha^{-1})$.

There are two ways of associating a weight to automorphic forms on G: One is to consider the cohomology group $H^d(\Gamma, L(\kappa; A))$ of an appropriate degree d for a given arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, and we call harmonic automorphic forms spanning $H^d(\Gamma, L(\kappa; \mathbb{C}))$ automorphic forms of (topological) weight κ. This way works well for any classical (or more general reductive) groups.

When the symmetric space of G is isomorphic to a (bounded) hermitian domain $\mathcal{H}$ with origin $\mathbf{0}$, like (the restriction of scalar to $\mathbb{Q}$ of) F-forms of Sp or $SU(m,n)$ over totally real fields F, we have another way to associate a weight to holomorphic automorphic forms. In this case, we have $\mathcal{H} \cong G(\mathbb{R})/C_{\mathbf{0}}$ for the stabilizer $C_{\mathbf{0}}$ of $\mathbf{0}$, which is a maximal compact subgroup of $G(\mathbb{R})$. In the simplest case of $SL(2)_{/\mathbb{Q}} = Sp(2)_{/\mathbb{Q}}$, $C_{\mathbf{0}} = SO_2(\mathbb{R})$ and $\mathcal{H} = H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with $G(\mathbb{R})/SO_2(\mathbb{R}) \cong H$ by $g \mapsto g(\sqrt{-1})$. As is well known that H is holomorphically equivalent to the open unit disk in $\mathbb{C}$ by $z \mapsto \frac{z-\sqrt{-1}}{z+\sqrt{-1}}$.

The group C_0 can be regarded as a group of real points with respect to a twisted complex conjugation in the complexification C of C_0. In the case of $SL(2)_{/\mathbb{Q}}$, $SO_2(\mathbb{R})$ can be regarded as S^1 in $\mathbb{G}_m(\mathbb{C})$ by $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \mapsto c\sqrt{-1}+d \in S^1$, and S^1 is the set of fixed points of the twisted "complex conjugation": $x \mapsto \overline{x}^{-1}$ in $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$. Generalizing this example, we see that the compact group $U(n)$ is the subgroup of $GL_n(\mathbb{C})$ fixed by the complex conjugation: $x \mapsto {}^t\overline{x}^{-1}$. Any holomorphic representation $\rho : C \to GL(V(\rho))$ gives rise to a holomorphic complex vector bundle $\widetilde{V} = (G(\mathbb{R}) \times V)/C_0$ by the action $(g, v) \mapsto (gu, u^{-1}v)$ for $u \in C_0$. Since $\mathcal{H}$ is simply connected, we can split $\widetilde{V} \cong \mathcal{H} \times V$ as holomorphic vector bundles; so, we have a linear map $J_\rho(g, z) : V_z \to V_{g(z)}$ for each given $g \in G(\mathbb{R})$ which identifies the fibers V_z and $V_{g(z)}$ of $\widetilde{V}$. Thus we have a function $J_\rho : G(\mathbb{R}) \times \mathcal{H} \to GL(V)$ satisfying

(1) (Cocycle Relation) $J_\rho(gh, z) = J_\rho(g, h(z))J_\rho(h, z)$ for $g, h \in G(\mathbb{R})$;

(2) (Holomorphy) $J_\rho(g, z)$ is holomorphic in z.

When $G = SL(2)$, then $C_0 = SO_2(\mathbb{R}) \subset C = \mathbb{C}^\times$ whose irreducible complex representation is given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \longmapsto \rho \begin{pmatrix} * & * \\ c & d \end{pmatrix} = (ci + d)^k = e^{ik\theta}.$$

In this case, $J_\rho(g, z) = (cz + d)^k$. This goes as follows: Split $GL_2(\mathbb{R}) = PC_0$ for P made of upper triangular matrices with right lower corner 1. For $z = x + iy$, define $p_z = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. Then for $g \in SL_2(\mathbb{R})$, write $gp_z = p_{g(z)}u$ with $u \in C_0$, and we have $\rho(u) = \rho(p_{g(z)}^{-1}gp_z) = (cz + d)^k$ by computation. Indeed, $J(g, z)$ sends (v, p_z) to $(uv, p_{g(z)}) \sim (v, gp_z) = (v, p_{g(z)}u)$.

One can view the complexification C as a real algebraic group; let T_C be a maximal real torus of C. To any character κ of T_C, we can attach a rational representation $L_C(\kappa; \mathbb{C})$ ($= \rho_\kappa$) of C. Let $V(\kappa) = L_C(\kappa; \mathbb{C})$. For an arithmetic discrete subgroup $\Gamma \subset G(\mathbb{Q})$, a holomorphic automorphic form of (coherent) weight κ is a holomorphic function $f : \mathcal{H} \to L_C(\kappa; \mathbb{C})$ satisfying $f(\gamma(z)) = J_\rho(\gamma, z)f(z)$ for all $\gamma \in \Gamma$ (with some additional growth condition if $\Gamma\backslash\mathcal{H}$ is not compact). Again the space of holomorphic automorphic forms is trivial unless the weight κ is positive (with respect to a fixed Borel subgroup B).

Often the complex manifold $\Gamma\backslash\mathcal{H}$ is canonically algebraizable, giving rise to an algebraic variety (or a scheme) X_Γ, called canonical models or *Shimura varieties*, defined over a valuation ring $\mathcal{W}$ in a number field with residual characteristic p. At the same time, we can algebraize the vector bundle $\widetilde{V}(\kappa)$ associated to $V(\kappa)$. Thus we often have a coherent sheaf $\underline{\omega}^\kappa$ on X_Γ giving rise to $\widetilde{V}(\kappa)$ after extending scalar to $\mathbb{C}$. The global sections of $H^0(X_\Gamma, \underline{\omega}^\kappa_{/A})$ for $\mathcal{W}$-algebra A are called A-integral automorphic forms of weight κ. Note that, T_C is isomorphic to T, because they are maximal tori in the same group G. Thus we can and will identify T and T_C (with compatible choice of Borel subgroups B and $B_C = B \cap C$). On X_Γ, we may regard the Γ-module $L_G(\kappa; A)$ as a locally constant sheaf associating to an open subset $U \subset X_\Gamma$

sections over U of the covering space $\widetilde{L}_G(\kappa; A) = \Gamma\backslash(\mathcal{D} \times L_G(\kappa; A))$ over X_Γ. Here the quotient $\Gamma\backslash(\mathcal{D} \times L_G(\kappa; A))$ is taken through the diagonal action. Thus each positive weight $\kappa \in X(T)$ gives two spaces of automorphic forms:

$$H^d(X_\Gamma, L_G(\kappa; A)), \quad H^0(X_\Gamma, \underline{\omega}^\kappa_{/A}) = G_\kappa(\Gamma; A).$$

There is (at least conjecturally) a correspondence $\kappa \mapsto \kappa^*$ such that

$$H^0(X_\Gamma, \underline{\omega}^\kappa) \hookrightarrow H^d(X_\Gamma, L_G(\kappa^*; \mathbb{C}))$$

by a "generalized Eichler-Shimura isomorphism" which is supposed to be equivariant under Hecke operators. If such equivariance holds, we say that the two modules: the source and the image are equivalent as *Hecke modules.* In the example of $SL(2)_{/\mathbb{Q}}$, we have $\kappa \in X(T) = X(\mathbb{G}_m) = \mathbb{Z}$ and $\kappa^* = \kappa - 2$ with:

$$G_\kappa(\Gamma; \mathbb{C}) \hookrightarrow H^1(X_\Gamma, L_{SL(2)}(\kappa - 2; \mathbb{C})) \quad (\Gamma \subset SL_2(\mathbb{Z}))$$

via $f \mapsto$ the cohomology class of $[f(z)(X - zY)^{\kappa-2}dz]$. This is valid if $\kappa \geqslant 2$.

1.2. p-Adic interpolation of automorphic forms. — We would like to interpolate these two sets of spaces $\{H^0(X_\Gamma, \underline{\omega}^\kappa)\}_\kappa$ and $\{H^d(X_\Gamma, L_G(\kappa; \mathcal{W}))\}_\kappa$ when the weights κ vary continuously in $\mathrm{Hom}_{top-gp}(T(\mathbb{Z}_p), \mathbb{Z}_p^\times)$. On these two spaces, there is a natural action of Hecke operators; so, we want this interpolation to take into account the Hecke operators. To describe our idea of how to interpolate automorphic forms, we write W for the p-adic completion of $\mathcal{W}$. What we would like to do in the two cases is:

(1) (Universality) Construct a (big) space V which is a compact module over $W[[T(\mathbb{Z}_p)]]$ such that the κ-eigenspace $V[\kappa]$ contains canonically the space $H^d(X_\Gamma, L_G(\kappa; W))$ in the topological case, resp. $H^0(X_{\Gamma/W}, \underline{\omega}^\kappa)$ in the coherent case as $W[[T(\mathbb{Z}_p)]]$-modules.

(2) (Hecke operators) Establish a natural action of Hecke operators on V, and show the inclusion in (1) is Hecke equivariant.

(3) (VCT) Find an appropriate $W[[T(\mathbb{Z}_p)]]$-submodule $X \subset V$ of co-finite type ($\Leftrightarrow$ W-dual is of finite type) such that X is stable under Hecke operators and $X[\kappa]$ is canonically isomorphic, as Hecke modules, to a well-described subspace of automorphic forms of weight κ if $\kappa \gg 0$.

The item (3) is called a *vertical control theorem* of the subspace X. Examples of the VCT are given as Theorem 3.2 for elliptic modular forms, Theorem 3.3 for p-adic family of elliptic modular forms, Theorem 8.5 for automorphic forms on unitary groups, Theorem 9.1 for Hilbert modular forms and Corollary 9.3 for Hilbert modular Hecke algebras. A more general result on VCT can be found in [**H02**] and [**PAF**]. In [**H02**] page 37 and [**GME**] 3.2.3, Hecke operators T are defined for a given (geometric) modular form f as a sum $f|T(A_{/S}) = \sum_\alpha f(A_{\alpha/S})$ of the values of f at abelian schemes A_α with a specific isogeny $\alpha : A \to A_\alpha$ of a given degree. This is perfectly fine if the degree is invertible on the base scheme S, but otherwise if S is of characteristic p