

AUTOUR DES MOTIFS

École d'été franco-asiatique de géométrie
algébrique et de théorie des nombres

*Asian-French summer school on algebraic
geometry and number theory*

Volume I

M. Kim, R. Sujatha,
L. Lafforgue,
A. Genestier, Ngô B. C.



Panoramas et Synthèses

Numéro 29

2009

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

PANORAMAS ET SYNTHÈSES 29

AUTOUR DES MOTIFS

École d'été franco-asiatique de géométrie algébrique
et de théorie des nombres

*Asian-French summer school on algebraic
geometry and number theory*

Volume I

M. Kim, R. Sujatha,
L. Lafforgue,
A. Genestier, Ngô B. C.

Société mathématique de France 2009

Publié avec le concours du Centre national de la recherche scientifique

Jean-Benoît Bost

Département de Mathématiques, Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud 11, 91405 Orsay Cedex, France

Jean-Marc Fontaine

Département de Mathématiques, Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud 11, 91405 Orsay Cedex, France

Alain Genestier

Institut de Mathématiques Élie Cartan U.M.R. 7502, Université Henri Poincaré Nancy 1. B.P. 70239, F-54506 Vandœuvre-lès-Nancy Cedex, France

E-mail : Alain.Genestier@iecn.u-nancy.fr

Minhyong Kim

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom

E-mail : minhyong.kim@ucl.ac.uk

Laurent Lafforgue

Institut des Hautes Études Scientifiques, 35 route de Chartres, F-91440 Bures-sur-Yvette, France

Bao Châu Ngô

Mathématiques, Bât. 425 Université Paris-Sud, 91405 Orsay France

E-mail : Bao-Chau.Ngo@math.u-psud.fr

Ramdorai Sujatha

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India

E-mail : sujatha@math.tifr.res.in

2000 Mathematics Subject Classification. — 11F, 11G, 11R39, 14F, 14G, 14K, 19E15.

Key words and phrases. — Motives, automorphic forms, Shimura varieties.

Mots-clé et phrases. — Motifs, formes automorphes, variétés de Shimura.



UNIVERSITÉ
PARIS-SUD 11

ASIAN-FRENCH SUMMER SCHOOL



IN ALGEBRAIC GEOMETRY AND NUMBER THEORY 2006 : MOTIVES AND RELATED TOPICS

JULY 17-29 2006

Ecole d'été franco-asiatique
de géométrie algébrique
et théorie des nombres 2006 :
autour des motifs

IHÉS, Bures-sur-Yvette
Centre de conférences
Marilyn et James Simons

REGISTRATION / INSCRIPTION

www.ihes.fr

LECTURES / COURS

Laurent LAFFORGUE

Formes automorphes et fonctorialité de Langlands

Marc LEVINE

Mixed motives and homotopy theory of schemes

NGÔ Báo Châu

Introduction to Shimura varieties

SAITO Takeshi

Galois representations and modular forms

AFTERNOON SEMINARS ORGANIZED BY / SÉMINAIRE DE L'APRÈS-MIDI ORGANISÉ PAR :

Yves ANDRÉ (CNRS, ENS Paris) - **Laurent CLOZEL** (Paris 11)

Philippe GILLE (CNRS, Paris 11) - **Michael HARRIS** (Paris 7)

Bruno KAHN (CNRS, Paris 7) - **Maxim KONTSEVICH** (IHÉS)

François LOESER (ENS Paris) - **Jörg WILDESHAUS** (Paris 13)

THIS SUMMER SCHOOL IS SUPPORTED BY THE FOLLOWING INSTITUTIONS / CETTE ÉCOLE BÉNÉFICIE DU SOUTIEN DES INSTITUTIONS SUIVANTES :

"Arithmetic Algebraic Geometry" European Network - Centre Sino-Français de Mathématiques - Chern Institute of Mathematics (Nankai University)

Institut Franco-Indien de Mathématiques - Institut Universitaire de France - Japan Society for the Promotion of Science

(programmes at Tokyo and Hiroshima Universities) - National Center for Theoretical Sciences (Taiwan) - National Science Council (Taiwan)

Service pour la Science et la Technologie de l'Ambassade de France en Chine - The Croucher Foundation (Hong Kong)

AUTOUR DES MOTIFS
**École d'été franco-asiatique de géométrie algébrique
et de théorie des nombres**
*Asian-French summer school on algebraic geometry
and number theory*
Volume I

M. Kim, R. Sujatha, L. Lafforgue, A. Genestier, Ngô B. C.

J.-B. Bost et J.-M. Fontaine, éditeurs

Abstract. — This volume contains the first part of the lecture notes of the *Asian-French summer school on algebraic geometry and number theory*, which was held at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and the université Paris-Sud XI (Orsay) in July 2006. This summer school was devoted to the theory of motives and its recent developments, and to related topics, notably Shimura varieties and automorphic representations.

The contributions in this first part are expanded versions of the talks introducing the theory of motives by M. Kim and R. Sujatha, the lecture notes *Quelques remarques sur le principe de fonctorialité* by L. Lafforgue, and *Lectures on Shimura varieties* by A. Genestier and Ngô B. C.

Résumé. — Ce volume contient la première partie des notes de cours de l'*École d'été franco-asiatique de géométrie algébrique et de théorie des nombres*, qui s'est tenue à l'Institut des Hautes Études Scientifiques et à l'université Paris-Sud XI en juillet 2006. Cette école était consacrée à la théorie des motifs et à ses récents développements, ainsi qu'à des sujets voisins, comme les théories des variétés de Shimura et des représentations automorphes.

Cette première partie est constituée de versions développées des exposés d'introduction à la théorie des motifs par R. Sujatha et M. Kim, puis des notes de cours *Quelques remarques sur le principe de fonctorialité* par L. Lafforgue et *Lectures on Shimura varieties* par A. Genestier et Ngô B. C.

TABLE DES MATIÈRES

Avant-Propos	ix
Foreword	xi
MINHYONG KIM — <i>Classical motives and motivic L-functions</i>	1
1. Motivation	1
2. Definitions	8
3. Conjectures on zeros, poles, and values	13
4. Remark	22
References	22
RAMDORAI SUJATHA — <i>Motives from a categorical point of view</i>	27
1. Introduction	27
2. Tannakian Categories	28
3. Category of motives	33
4. Motives and Tannakian categories	35
5. Motivic Galois groups	39
6. Motives over finite fields	42
7. Mixed Motives and Algebraic K -theory	43
References	47
LAURENT LAFFORGUE — <i>Quelques remarques sur le principe de fonctorialité</i> .	49
Introduction	49
Exposé I : Principe de diagonalisation	52
Exposé II : Travail sur les noyaux	75
Exposé III : Développement asymptotiques	99
Exposé IV : Et du côté géométrique ?	133
Exposé V : Des calculs géométriques et une conjecture dans le cas de l'induction automorphe de GL_1 à GL_2 ou GL_r	152
Références	184
ALAIN GENESTIER & BAO CHÂU NGÔ — <i>Lectures on Shimura varieties</i>	187
1. Quotients of Siegel's upper half space	187

2. The moduli space of polarized abelian schemes	194
3. Shimura varieties of PEL type	205
4. Shimura varieties	213
5. CM tori and canonical models	221
6. Points of Siegel varieties over a finite field	229
References	235

AVANT-PROPOS

Ce volume contient la première partie des notes de cours de l'école d'été franco-asiatique de géométrie algébrique et de théorie des nombres *Autour des motifs*, qui s'est tenue à l'Institut des Hautes Études Scientifiques et à l'Université Paris-Sud XI en Juillet 2006.

Au nom des membres français du comité d'organisation de l'école d'été, nous voulons exprimer toute notre gratitude aux collègues étrangers qui ont accepté de se joindre à ce comité : leur contribution aux multiples tâches nécessaires à l'organisation d'une telle école — notamment l'obtention de financements, la sélection des étudiants et la mise au point du programme scientifique — a été déterminante.

Nous adressons nos vifs remerciements à Marie-Claude Vergne pour la réalisation de l'affiche de l'école d'été, reproduite en page iv.

L'édition de ce volume a bénéficié des relectures attentives et des suggestions de Laurent Clozel, Laurent Fargues, Fu Lei et Joël Riou. Nous les en remercions chaleureusement.

Orsay, Juillet 2009

Jean-Benoît Bost et Jean-Marc Fontaine

FOREWORD

The *Asian-French Summer School on Algebraic Geometry and Number Theory* was devoted to the theory of motives and its recent developments, and to related topics, notably Shimura varieties and automorphic representations. It was held at the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, and the Université Paris-Sud XI, Orsay, from July 17 to July 29, 2006. The Summer School, although open to all, was more specifically aimed towards doctoral and post-doctoral students. Its purpose was to make accessible some recent progresses in algebraic geometry and number theory, and to stimulate contacts between young researchers from Asia and Europe.

The Organizing Committee consisted of Jean-Benoît Bost, Jean Pierre Bourguignon, Jean-Marc Fontaine, Fu Lei, Laurent Lafforgue, Minhyong Kim, Marc Levine, Ngaiming Mok, Ngô Bảo Châu, Takeshi Saito, Sujatha Ramdurai, and Chia-Fu Yu.

Beside the host institutions (IHÉS and Paris-Sud XI), the Summer School benefited from the support of the following organizations: “Arithmetic Algebraic Geometry” European Network, Brain Korea 21, Centre Sino-Français de Mathématiques, Chern Institute of Mathematics (Nankai University), Institut Franco-Indien de Mathématiques, Institut Universitaire de France, Korea Institute for Advanced Study, Japan Society for the Promotion of Science, National Center for Theoretical Sciences (Taiwan), National Science Council (Taiwan), Service pour la Science et la Technologie de l’Ambassade de France en Chine, The Croucher Foundation (Hong Kong).

The program of the Summer School firstly consisted of four series of lectures, scheduled in the morning:

- L. Lafforgue, *Formes automorphes et fonctorialité de Langlands*,
- M. Levine, *Mixed motives and homotopy theory of schemes*,
- Ngô B. C., *Introduction to Shimura varieties*,
- T. Saito, *Galois representations and modular forms*.

In addition, a series of afternoon seminars was organized, devoted to various themes, either supplementing the material discussed in the morning lectures, or providing introductions to some other aspects of the theory of motives. In chronological order, the following topics were discussed during these seminars:

1. *An introduction to “classical” motives*, by M. Kim and R. Sujatha,
2. *Motives and motivic integration for definable sets*, by F. Loeser and R. Cluckers,

3. *Automorphic representations, motives, and Galois representations, I–II–III* by L. Fargues, L. Clozel, P.-H. Chaudouard, and M. Harris,
4. *Quadratic forms and algebraic cycles*, by Ph. Gille and N. Karpenko,
5. *Periods and motivic Galois groups* and *Motives of non-commutative spaces*, by M. Kontsevich,
6. *Realization functors and the full faithfulness conjectures*, by J. Riou and Y. André,
7. *Complements on categories of motives*, by B. Kahn and F. Ivorra,
8. *Mixed motives and Shimura varieties*, by J. Wildeshaus and D.-C. Cisinsky.

The contributions in this volume are expanded versions of the “introduction” seminars by M. Kim and R. Sujatha and of the lectures by L. Lafforgue and Ngô B. C.

The second volume will contain the notes of the lectures by M. Levine and T. Saito and of seminars 3, 4, 6, 7, 8 above.

CLASSICAL MOTIVES AND MOTIVIC L-FUNCTIONS

by

Minhyong Kim

The exposition here follows the lecture delivered at the summer school, and hence, contains neither precision, breadth of comprehension, nor depth of insight. The goal rather is the curious one of providing a loose introduction to the excellent introductions that already exist, together with scattered parenthetical commentary. The inadequate nature of the exposition is certainly worst in the third section. As a remedy, the article of Schneider [39] is recommended as a good starting point for the complete novice, and that of Nekovář [36] might be consulted for more streamlined formalism. For the Bloch-Kato conjectures, the paper of Fontaine and Perrin-Riou [19] contains a very systematic treatment, while Kato [26] is certainly hard to surpass for inspiration. Kings [29], on the other hand, gives a nice summary of results (up to 2003).

1. Motivation

Given a variety X over $\mathbb{Q}$, it is hoped that a suitable analytic function

$$\zeta(X, s),$$

a ζ -function of X , encodes important arithmetic invariants of X . The terminology of course stems from the fundamental function

$$\zeta(\mathbb{Q}, s) = \sum_{n=1}^{\infty} n^{-s}$$

named by Riemann, which is interpreted in this general context as the zeta function of $\mathrm{Spec}(\mathbb{Q})$. A general zeta function should generalize Riemann's function in a manner similar to Dedekind's extension to number fields. Recall that the latter can be defined by replacing the sum over positive integers by a sum over ideals:

$$\zeta(F, s) = \sum_I N(I)^{-s}$$

where I runs over the non-zero ideals of the ring of integers $\mathcal{O}_F$ and $N(I) = |\mathcal{O}_F/I|$, and that $\zeta(F, s)$ has a simple pole at $s = 1$ (corresponding to the trivial motive factor of $\text{Spec}(F)$, as it turns out) with

$$(s-1)\zeta(F, s)|_{s=1} = \frac{2^{r_1}(2\pi)^{r_2}h_F R_F}{w_F \sqrt{|D_F|}}$$

By the unique factorization of ideals, $\zeta(F, s)$ can also be written as an Euler product

$$\prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1}$$

as $\mathcal{P}$ runs over the maximal ideals of $\mathcal{O}_F$, that is, the closed points of $\text{Spec}(\mathcal{O}_F)$. Now, if a scheme $\mathcal{Y}$ is of finite type over $\mathbb{Z}$, then for any closed point $y \in \mathcal{Y}$, its residue field $k(y)$ is finite. Write $N(y) := |k(y)|$. We can then form an Euler product [40]

$$Z(\mathcal{Y}, s) := \prod_{y \in \mathcal{Y}_0} (1 - N(y)^{-s})^{-1},$$

where $(\cdot)_0$ denotes the set of closed points for any scheme $(\cdot)$. In the case when the map

$$\mathcal{Y} \rightarrow \text{Spec}(\mathbb{Z})$$

factors through $\text{Spec}(\mathbb{F}_p)$, $Z(\mathcal{Y}, s)$ reduces to Weil's zeta function for a variety over a finite field (with the substitution $p^{-s} \mapsto t$ if a formal variable has intervened as in [40], section 1.6).

When we are starting with $X/\mathbb{Q}$, which we assume throughout to be proper and smooth, a straightforward imitation of Dedekind's definition might involve taking an integral model $\mathcal{X}$ of X , which is a proper flat scheme of finite-type over $\mathbb{Z}$ with X as generic fiber, and defining

$$\zeta(X, s) \text{ ``} := \text{''} Z(\mathcal{X}, s) = \prod_{x \in \mathcal{X}_0} (1 - N(x)^{-s})^{-1}$$

The problem with this approach is that the function thus obtained will depend on the model, and there is no general method for choosing a canonical one. However, there will be some set S of primes such that there is a model $\mathcal{X}_S$ over $\text{Spec}(\mathbb{Z}[1/S])$ which is furthermore *smooth*. Even though such a $\mathbb{Z}[1/S]$ -model need be no more canonical, it does turn out that the incomplete zeta function

$$\zeta_S(X, s) := \prod_{x \in (\mathcal{X}_S)_0} (1 - N(x)^{-s})^{-1}$$

is independent of the model. (More on this point below.) So there are good elementary generalizations of incomplete zeta functions. We note in this connection that

$$Z(\mathcal{X}, s) = \prod_p Z(\mathcal{X}_p, s)$$

where

$$\mathcal{X}_p = \mathcal{X} \otimes \mathbb{F}_p$$