Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

A SPECIAL DEBARRE–VOISIN FOURFOLD

Jieao Song

Tome 151
Fascicule 2

2023

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 305-330

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 2, tome 151, juin 2023

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
151 (2), 2023, p. 305-330

A SPECIAL DEBARRE–VOISIN FOURFOLD

by Jieao Song

Abstract. — Consider the finite simple group $\mathbf{G} \coloneqq \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660, which has an irreducible representation V_{10} of dimension 10. In this note, we study a special trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ that is $\mathbf{G}$-invariant. Following the construction of Debarre–Voisin, we obtain a smooth hyperkähler fourfold $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ with many symmetries. We will also look at the associated Peskine variety $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$, which is highly symmetric as well and admits 55 isolated singular points. It will help us to better understand the geometry of the special Debarre–Voisin fourfold $X_6^{\sigma_0}$. We also discuss an application of this example to the global geometry of the moduli space of Debarre–Voisin fourfolds.

Résumé (*Une variété de Debarre–Voisin spéciale*). — Considérons le groupe simple fini $\mathbf{G} \coloneqq \mathrm{PSL}(2, \mathbf{F}_{11})$ d'ordre 660, qui admet une représentation irréductible V_{10} de dimension 10. Nous allons étudier un trivecteur $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ qui est $\mathbf{G}$-invariant. En suivant la construction de Debarre–Voisin, nous obtenons une variété hyperkählérienne $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ lisse de dimension 4 avec beaucoup de symétries. Nous allons aussi étudier la variété de Peskine associée $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$, qui admet 55 points singuliers isolés et est également très symétrique. Cette dernière nous permet de mieux comprendre la géométrie de la variété spéciale $X_6^{\sigma_0}$. Nous discuterons aussi d'une application de cet exemple à la géométrie globale de l'espace de modules des variétés de Debarre–Voisin.

Texte reçu le 1er octobre 2021, modifié le 28 novembre 2022, accepté le 14 décembre 2022.

Jieao Song, Université Paris Cité, CNRS, IMJ-PRG, 75013 Paris, France • *E-mail :* `jieao.song@imj-prg.fr`

Mathematical subject classification (2010). — 14J35, 14J50.

Key words and phrases. — Hyperkähler manifolds, automorphisms of varieties.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2023/305/$ 5.00

doi:10.24033/bsmf.2870

1. Introduction

The study of automorphism groups for K3 surfaces and higher dimensional hyperkähler manifolds is a rich subject that has many deep relations with lattice theory and representation theory of simple groups. For example, in [13], Mukai showed that a finite group of symplectic automorphisms of a K3 surface is always a subgroup of the Mathieu group M_{23}. Similarly, in [12, Theorem 7.2.4], Mongardi showed that a finite group of symplectic automorphisms of a hyperkähler manifold of $\mathrm{K3}^{[2]}$-type is a subgroup of the Conway group Co_1. Moreover, for a such manifold X, the maximal prime order of any symplectic automorphism is 11, and in this case, X must have maximal Picard rank 21, so it is isolated in the moduli.

Consider the finite simple group $\mathbf{G} := \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660. Mongardi constructed a special cubic fourfold, as well as a special Eisenbud–Popescu–Walter sextic with a faithful $\mathbf{G}$-action. From these, one obtains two hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type—the corresponding Fano variety of lines and double EPW sextic—that are highly symmetric (see [12, Section 4.5] and [7]). We also note that a complete classification of symplectic automorphism groups for cubic fourfolds is available in [10].

In this paper, we study an explicit example of a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type in the Debarre–Voisin family that also admits a faithful $\mathbf{G}$-action. A key feature of this example is that we can describe explicitly its Picard lattice using the geometry of some associated Fano varieties.

Let V_{10} be a 10-dimensional complex vector space. A *Debarre–Voisin variety* X_6^σ is defined inside the Grassmannian $\mathrm{Gr}(6, V_{10})$ from the datum of a trivector $\sigma \in \bigwedge^3 V_{10}^\vee$. By studying the representations of the group $\mathbf{G}$, it is not hard to find a candidate for the special trivector σ_0: denote by V_{10} one of the two 10-dimensional irreducible representations of $\mathbf{G}$; there exists a unique (up to multiplication by a scalar) trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ that is $\mathbf{G}$-invariant.

Using the general results obtained in [2] on the geometry of Debarre–Voisin varieties and associated Peskine varieties, one can study in detail the geometry of this special Debarre–Voisin variety $X_6^{\sigma_0}$. We prove the following results.

Theorem 1.1. — *Let $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ be the special $\mathbf{G}$-invariant trivector.*

1. *(Proposition 3.2) The Debarre–Voisin variety $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ is smooth of dimension* 4.
2. *(Proposition 4.2) The associated Peskine variety $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$ has* 55 *isolated singular points. The group $\mathbf{G}$ acts transitively on them.*
3. *(Proposition 5.6) The group $\mathrm{Aut}_H^s(X_6^{\sigma_0})$ of symplectic automorphisms that fix the polarization H on $X_6^{\sigma_0}$ is isomorphic to $\mathbf{G}$.*
4. *One can give an explicit description of the Picard lattice of $X_6^{\sigma_0}$, which has maximal rank* 21. *It is spanned by* 55 (-2)*-classes (see* (4) *for the*

Gram matrix). Moreover, if we denote by $H^2_{\mathrm{trans}}(X_6^{\sigma_0})$ *the transcendental lattice and by* $T := H^2(X_6^{\sigma_0}, \mathbf{Z})^{\mathbf{G}}$ *the* $\mathbf{G}$*-invariant sublattice, we have the following isomorphisms of lattices (Proposition 5.9)*

$$H^2_{\mathrm{trans}}(X_6^{\sigma_0}) \simeq L_{11} := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 6 \end{smallmatrix}\right), \qquad T = H^2_{\mathrm{trans}}(X_6^{\sigma_0}) \oplus \langle H \rangle \simeq L_{11} \oplus (22),$$
$$\mathrm{Pic}(X_6^{\sigma_0}) \simeq U \oplus E_8(-1)^{\oplus 2} \oplus L(-1),$$

where the component L *can be taken to be both* $\left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 8 \end{smallmatrix}\right)$ *and* $L_{11} \oplus (2)$.

5. *(Proposition 5.12)* $X_6^{\sigma_0}$ *can be characterized as the unique Debarre–Voisin fourfold admitting a symplectic automorphism of order* 11.
6. $X_6^{\sigma_0}$ *is birationally isomorphic to the Hilbert square of a K3 surface (Proposition 5.14) and is special in the sense of Hassett for all possible discriminants* $d \geq 24$ *(Proposition 5.15).*

The property of being Hassett special for all possible discriminants $d \geq 24$ has a nice implication on the global geometry of the moduli space of Debarre–Voisin fourfolds. Namely, we have two different moduli spaces in this setting: the GIT moduli space $\mathcal{M}_{\mathrm{DV}}$ of trivectors and the moduli space $\mathcal{M}_{22}^{(2)}$ of polarized hyperkähler manifolds. The Debarre–Voisin construction provides a rational map

$$\mathfrak{m}\colon \mathcal{M}_{\mathrm{DV}} \dashrightarrow \mathcal{M}_{22}^{(2)},$$

which is proved to be birational [16, Theorem 1.8]. Moreover, one can show that the restriction of $\mathfrak{m}$ to the open locus $\mathcal{M}_{\mathrm{DV}}^{\mathrm{sm}}$ of trivectors defining a smooth Debarre–Voisin fourfold is an open immersion (Proposition 6.3).

When we resolve the indeterminacies of this map, the image of each exceptional divisor is called *Hassett–Looijenga–Shah* (HLS) (see Definition 6.4), which reflects a difference between the GIT and the Baily–Borel compactifications. The result on $X_6^{\sigma_0}$ implies that all Heegner divisors $\mathcal{D}_d$ for $d \geq 24$ are not HLS (Corollary 6.5). Combined with the results of [5] and [15], one concludes that a Heegner divisor $\mathcal{D}_d$ is HLS if and only if $d \in \{2, 6, 8, 10, 18\}$. We discuss this in Section 6.

Notation. — We use σ to denote a trivector and σ_0 to denote the special $\mathbf{G}$-invariant trivector.

2. The special trivector

We first give the construction of the special trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$.

The finite simple group $\mathbf{G} := \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660 admits eight different irreducible complex representations: two of them are of dimension 5 and will be denoted by V_5 and $V_5^\vee$. They are the dual to each other.

A classical result is that the symmetric power $\mathrm{Sym}^3 V_5^\vee$—the space of cubic polynomials on V_5—admits an irreducible subrepresentation of dimension 1:

for a suitable choice of basis $(y_0, \dots, y_4)$ of $V_5^\vee$, this corresponds to the Klein cubic with equation

$$y_0^2 y_1 + y_1^2 y_2 + y_2^2 y_3 + y_3^2 y_4 + y_4^2 y_0 \in \mathrm{Sym}^3 V_5^\vee.$$

In [1], Adler showed that the automorphism group of this smooth cubic is precisely the group $\mathbf{G}$.

The wedge product $\bigwedge^2 V_5$ gives another irreducible representation, of dimension 10, which is self-dual and will be denoted by V_{10}. We consider elements of $\bigwedge^3 V_{10}^\vee$. A computation of characters tells us that this representation of $\mathbf{G}$ also admits one irreducible subrepresentation of dimension 1, generated by a $\mathbf{G}$-invariant trivector σ_0. The characters of all eight irreducible representations of $\mathbf{G}$ as well as the character of $\bigwedge^3 V_{10}^\vee$ can be found in Section B, Table B.1. Note that the other irreducible representation V_{10}' of dimension 10 does not provide $\mathbf{G}$-invariant trivectors (see also Remark 5.13 on the uniqueness of the trivector σ_0).

We now give a concrete description of the special trivector σ_0 in terms of coordinates in a suitable basis. The subgroup $\mathbf{B}$ of $\mathbf{G}$ of upper triangular matrices can be generated by the elements

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$$

of the respective orders 11 and 5. Write $\zeta = e^{2\pi i/11}$ and $\rho\colon \mathbf{G} \to \mathrm{GL}(V_5^\vee)$ for the representation $V_5^\vee$. In a suitable basis $(y_0, \dots, y_4)$ of $V_5^\vee$, the matrices of P and R are

$$\rho(P) = \begin{pmatrix} \zeta^1 & 0 & 0 & 0 & 0 \\ 0 & \zeta^9 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^5 \end{pmatrix} \quad \text{and} \quad \rho(R) = \begin{pmatrix} 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}. \tag{1}$$

Note that one can already identify the equation of the $\mathbf{G}$-invariant Klein cubic using only these two elements, instead of the whole group $\mathbf{G}$.

The elements $y_{ij} := y_i \wedge y_j$ form a basis of $V_{10}^\vee$. In this basis, we see that P acts diagonally and R as a permutation (see Table 2.1; note that we have chosen a particular order in which the action of R is very simple). We may easily verify that the space of trivectors invariant under the action of P and R is of dimension 2 and is spanned by the $\mathbf{B}$-invariant trivectors

$$\begin{aligned} \sigma_1 &:= y_{01} \wedge y_{23} \wedge y_{02} + y_{12} \wedge y_{34} \wedge y_{13} + y_{23} \wedge y_{40} \wedge y_{24} \\ &\quad + y_{34} \wedge y_{01} \wedge y_{30} + y_{40} \wedge y_{12} \wedge y_{41}, \\ \sigma_2 &:= y_{01} \wedge y_{41} \wedge y_{24} + y_{12} \wedge y_{02} \wedge y_{30} + y_{23} \wedge y_{13} \wedge y_{41} \\ &\quad + y_{34} \wedge y_{24} \wedge y_{02} + y_{40} \wedge y_{30} \wedge y_{13}. \end{aligned}$$

TABLE 2.1. The action of P and R in the basis (y_{ij})

	y_{01}	y_{12}	y_{23}	y_{34}	y_{40}	y_{02}	y_{13}	y_{24}	y_{30}	y_{41}
Eigenvalues of $\bigwedge^2 \rho(P)$	ζ^{10}	ζ^2	ζ^7	ζ^8	ζ^6	ζ^5	ζ^1	ζ^9	ζ^4	ζ^3
Action of $\bigwedge^2 \rho(R)$	y_{12}	y_{23}	y_{34}	y_{40}	y_{01}	y_{13}	y_{24}	y_{30}	y_{41}	y_{02}

To identify the unique $\mathbf{G}$-invariant trivector, we must look at some elements in $\mathbf{G} \smallsetminus \mathbf{B}$. Since the explicit description for the representation V_5 is known [18], we will pick one such element and compute its matrix explicitly.

The group $\mathbf{G}$ admits a presentation with two generators a, b and relations $a^2 = b^3 = (ab)^{11} = [a, babab]^2 = 1$. We can take $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. One may check that $ab = P$ while $bbababab bababab b = R$. Matrices for $\rho(a)$ and $\rho(b)$ are provided by [18], so the representation is completely determined. Choose a suitable basis of $V_5^\vee$ consisting of eigenvectors of $\rho(ab) = \rho(P)$. In this basis, the matrices of P and R are as in (1). Since the element a does not lie in the subgroup $\mathbf{B}$, we use its matrix in this new basis to verify that the unique (up to multiplication by a scalar) $\mathbf{G}$-invariant trivector is $\sigma_0 := \sigma_1 + \sigma_2$.

From now on, we will rewrite the basis (y_{ij}) as $(x_0, \ldots, x_9)$ in the order chosen in Table 2.1, so the trivector σ_0 is given by

$$\begin{aligned} \sigma_0 = {} & x_0 \wedge x_2 \wedge x_5 + x_1 \wedge x_3 \wedge x_6 + x_2 \wedge x_4 \wedge x_7 + x_3 \wedge x_0 \wedge x_8 + x_4 \wedge x_1 \wedge x_9 \\ & + x_0 \wedge x_9 \wedge x_7 + x_1 \wedge x_5 \wedge x_8 + x_2 \wedge x_6 \wedge x_9 + x_3 \wedge x_7 \wedge x_5 + x_4 \wedge x_8 \wedge x_6, \end{aligned}$$

or more succinctly,

$$\begin{aligned} \sigma_0 = {} & [025] + [136] + [247] + [308] + [419] \\ & + [097] + [158] + [269] + [375] + [486]. \end{aligned} \tag{2}$$

We have, therefore, shown the following result.

PROPOSITION 2.1. — *Up to multiplication by a scalar, the trivector σ_0 in* (2) *is the unique $\mathbf{G}$-invariant trivector in $\bigwedge^3 V_{10}^\vee$, where V_{10} is the* 10*-dimensional irreducible $\mathbf{G}$-representation given in Table B.1.*

3. The Debarre–Voisin fourfold

The Debarre–Voisin variety associated with a non-zero trivector σ is the scheme

$$X_6^\sigma := \{[V_6] \in \mathrm{Gr}(6, V_{10}) \mid \sigma|_{V_6} = 0\}$$

in the Grassmannian $\mathrm{Gr}(6, V_{10})$ parametrizing those $[V_6]$ on which σ vanishes. Its expected dimension is 4. For σ general, it is shown in [8] that X_6^σ is a smooth