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GINZBURG-LANDAU VORTICES: THE STATIC MODEL

by **Tristan RIVIÈRE**

1. INTRODUCTION

1.1. Physical origin of the problem

One of the first explanatory models for superconductivity (which refers to the existence of permanent currents in certain substances, with no energy dissipation) has been proposed during the fifties by V. Ginzburg and L. Landau, from the Landau theory of phase transitions. Following this model, the degree of superconductivity of a body occupying a domain Ω of $\mathbb{R}^3$, is characterized by a "wave function" $\Omega \to \mathbb{C}$ referred to as the order parameter. In the quantum theory of J. Bardeen, L.N. Cooper and J. Schrieffer (BCS theory), which came in 1957 to justify the Ginzburg-Landau phenomenological model, the square of the modulus of this order parameter $|u|^2$ represents the local electron pair (Cooper pairs) density, responsible for the superconductivity. For $|u| = 1$ this density is maximum and minimum for $|u| = 0$.

The energy functional for a superconductor proposed by Ginzburg and Landau is

$$\mathcal{J}(u, A) = \int_\Omega |\kappa^{-1}du - iAu|^2 + \frac{1}{2}|1 - |u|^2|^2 + |dA|^2 - 2\int_\Omega dA.h_e$$

where A is the 1-form vector potential associated to the induced field dA in the superconductor ($du - iAu$ is thus a 1-form taking its values in $\mathbb{C}$). h_e is the 2-form representing the external field applied to the superconductor. This is one of the parameters of the problem together with the constant κ, known as the coupling constant, which depends on the sample considered, and which plays an essential part in the theory, as we shall see in the following. As a ratio of two lengths, $\kappa = \frac{\lambda}{\xi}$, where λ is the penetration depth of the external field h_e in the sample (see the following) and ξ is the characteristic size of a vortex (see section 2), is a dimensional constant. Note that this functional is also the Yang-Mills-Higgs action in the abelian gauge theory modeling the interaction between a classical magnetic field and a Higgs particle.

Schematically, the observed phenomena are as follows. When the applied field is zero, the superconductor is said to be in the *pure state*:

$$\begin{cases} |u| = 1 & \text{in } \Omega \\ dA = 0 & \text{in } \Omega. \end{cases}$$

The density of Cooper pairs is maximum and the induced field is zero. When the applied field is sufficiently strong (sample dependent) the superconductivity disappears:

$$\begin{cases} |u| = 0 & \text{in } \Omega \\ dA = h_e & \text{in } \Omega. \end{cases}$$

The density of Cooper pairs is then minimal and the induced field coincides with the applied field. The superconductor is in the *normal state.*

The nature of the transition from the *pure state* to the *normal state* depends on the composite one and in particular on the value of κ. One observes that for $\kappa < 1/\sqrt{2}$ (type I superconductor), this transition is sharp and happens for a certain strength of the applied field which is independent of κ. Instead, for $\kappa > 1/\sqrt{2}$ (type II superconductor), as the external field increases, to go from the *pure state* to the *normal state*, we pass through a different phase known as a *mixed state*, where more and more regions of normal state contained in tubes (vorticity filaments) around which the phase of u makes one or several circular turns, appears. When the sample is homogeneous and the external field uniform, these tubes line up in the direction of the field, to form periodic Abrikosov lattices, named after the physician who first showed their existence. It is observed that this lattice is triangular in the fundamental state. We pass from the *pure state* to the *mixed state*, for an applied field known as the "first critical field" $H_{c_1} \simeq O\left(\frac{\log \kappa}{\kappa}\right)$, and we leave the *mixed state* to go into the *normal state* for an applied field known as the "second critical field" $H_{c_2} \simeq O(\kappa)$. The phase diagram (figure 1) summarizes the observations mentioned above. For a more complete account of the physics of superconductors the reader can refer to: [dG], [SST], [Ti]...

1.2. The mathematical questions underlying superconductivity

There are numerous difficulties that arise when one wants to give a mathematically rigorous sense to the previous observations, starting from the Ginzburg-Landau model. A first reduction is to consider a 1 or 2-dimensional version of the model ($\Omega \subset \mathbb{R}^3$ and h_e then have the symmetries corresponding to those reductions: space comprised between two parallel planes or infinite cylinder, in uniform magnetic fields...). In this talk, we shall not consider the studies in 1-dimension which are however extensive and which enable very often a more refined analysis of the phase diagram ([BH1], [BH2], [Af]... for a complete presentation of these results, see [AT]). We shall consider here only the 2-dimensional case, which is the minimal dimension to observe vortices (dimension 3 and higher dimensions are treated in [Ri2], [LR], [LR2] and also in [BBM]).

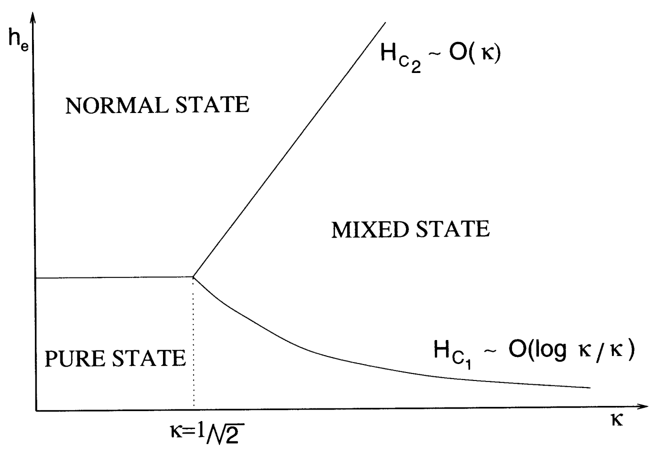

FIGURE 1. Phases diagram

Ω is thus an infinite cylinder and h_e is a uniform field parallel to the direction of the axis of the cylinder. Ω then denotes the 2-dimensional section of this cylinder and h_e being a 2-form which is constant on this section is often confused with the number giving its intensity. The aim is thus to understand the nature of fundamental states of the functional $\mathcal{J}$, and also of the critical points in general, as a function of the different values of (κ, h_e) in the phase diagram represented in figure 1. By "understanding the nature of the fundamental states of $\mathcal{J}$", we mean essentially identifying the zero set of the order parameter u of a solution minimizing $\mathcal{J}$, which corresponds to the 2-dimensional section of the vortex lattice expected in the mixed phase.

To simplify the analysis we consider the change of variables $A \longrightarrow \kappa A$ in the original model, which then leads us to the functional

$$\mathcal{G}_\kappa(u, A) = \int_\Omega |du - iAu|^2 + \frac{\kappa^2}{2}|1 - |u|^2|^2 + |dA|^2 + 2h_e \int_\Omega dA.$$

This new functional verifies the gauge invariance $\mathcal{G}_\kappa(u, A) = \mathcal{G}_\kappa(e^{i\phi}u, A + d\phi)$ for any function ϕ on Ω. It is then possible to extend the model to any domain Ω which is any 2-dimensional manifold. (u, A) are then respectively the sections and connections of a complex line bundle E on Ω on which we fix a hermitian product whose real part is $(\ ,\ )$ or $|\ |^2$ for the quadratic form. $du - iAu$ is replaced by the covariant derivative $d_A u$ of u with respect to A and dA is the curvature of the connection A. In the following, we note $h = *dA$.

Section 2 is devoted to the study of the Ginzburg-Landau free energy $\mathcal{F}$ without interaction with the external field (i.e $h_e = 0$). In 2.1, we present the work of Jaffe and Taubes on the integrable or non-interacting case $\kappa = 1/\sqrt{2}$. We state their conjectures for the cases $\kappa < 1/\sqrt{2}$ and $\kappa > 1/\sqrt{2}$. In 2.2, we describe the BBH asymptotic analysis (F. Bethuel, H. Brezis and F. Hélein [BBH]) in the London limit: $\kappa \to +\infty$ which corresponds to the strongly repulsive case. In this limit, to which we shall restrict in the following, the vorticity phenomena appear more clearly; this case is also close to many of the type II superconductors we have in practice, for which the a-dimensional parameter κ is very large. In section 2.3, we revisit the part played by the renormalized energy $\mathcal{W}$ coming from the BBH asymptotic analysis used to describe the critical points of $\mathcal{F}$. Finally, we give answers to the conjectures of Jaffe and Taubes in the London limit and we extend them to more general cases. The third part is devoted to the complete study of the functional $\mathcal{G}$ comprising the interaction term with the external field. The vorticity is then no longer a fixed parameter as in the previous section but becomes a variable of the problem. The contents of this section covers part of S. Serfaty's PhD thesis, and the work she did in collaboration with E. Sandier.

2. STUDY OF THE FREE ENERGY FUNCTIONAL $\mathcal{F}$

2.1. The integrable or non-interacting case $\kappa = 1/\sqrt{2}$

In [JT], A. Jaffe and C. Taubes study the critical points on $\mathbb{R}^2$ of the free energy functional

$$\mathcal{F}_\kappa(u, A) = \int_{\mathbb{R}^2} |d_A u|^2 + \frac{\kappa^2}{2}|1 - |u|^2|^2 + |dA|^2$$

and which are solutions to the Euler equations

$$\left\{\begin{array}{l} d_A^* d_A u = \kappa^2 u(1 - |u|^2) \\ \\ d^* dA = (iu, d_A u) \end{array}\right. \tag{1}$$

where d_A^* is the operator acting on the 1-forms η such as $d_A^*\eta = d^*\eta + iA\wedge\eta$. Supposing that the intrinsic quantities $|d_A u|$, $|1 - |u||$ and dA are decreasing (polynomially), the renormalized magnetic field is an integer N

$$N = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA$$

which corresponds to the degree of $u/|u|$ on circles of sufficiently large radii. This is known as the homotopy class of the couple (u, A). For a given N, say $N \geq 0$, it has been observed by E.B. Bogomol'nyi [Bog] that, for the particular value $\kappa = 1/\sqrt{2}$, the functional $\mathcal{F}$ can be rewritten under the following form

$$\begin{aligned}\mathcal{F}_\kappa(u, A) \int_{\mathbb{R}^2} \quad & |\Re(d_{A_1} u) - \Im(d_{A_2} u)|^2 + |\Re(d_{A_2} u) - \Im(d_{A_1} u)|^2 \\ & + | * dA + \tfrac{1}{2}(|u|^2 - 1)|^2 + 2\pi N\end{aligned}$$