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SYMPLECTIC LOCAL ROOT NUMBERS, CENTRAL CRITICAL L-VALUES, AND RESTRICTION PROBLEMS IN THE REPRESENTATION THEORY OF CLASSICAL GROUPS

by

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Abstract. — In this paper, we provide a conjectural recipe for the restriction of irreducible representations of classical groups (including metaplectic groups), to certain subgroups, generalizing our earlier work on representations of orthogonal groups. Our conjectures include the cases of Bessel and Fourier-Jacobi models. In fact, it is the standard representation of the classical group, together with its orthogonal, symplectic, hermitian, or skew-hermitian form, that plays the primary role, and not the classical group alone. All of our conjectures assume the Langlands parametrization. For classical groups over local fields, the recipe involves local epsilon factors associated to the Langlands parameter and certain summands of a fixed symplectic representation of the L-group. For automorphic representations over global fields, it involves the central critical value of this symplectic L-function.

***Résumé* (Nombres de racines locales symplectiques, L-valeurs critiques centrales et problèmes de restriction en théorie de représentation des groupes classiques)**

Dans cet article, nous donnons une recette conjecturale pour la restriction à certains sous-groupes des représentations irréductibles de groupes classiques. Cela inclut les groupes métaplectiques et généralise notre travail antérieur pour les groupes orthogonaux. Nos conjectures comprennent les cas des modèles de Bessel et Fourier-Jacobi. En fait le rôle principal est joué, non par le groupe seul, mais par la représentation naturelle de ce groupe classique, munie de sa forme bilinéaire-orthogonale, symplectique, hermitienne ou anti-hermitienne selon le cas. Dans toutes nos conjectures, nous admettons que la paramétrisation de Langlands est établie. Notre recette, pour les groupes classiques sur les corps locaux, fait intervenir les facteurs epsilon locaux associés au paramètre de Langlands et certains facteurs d'une représentation symplectique fixée du L-groupe. Pour les représentations automorphes sur des corps globaux, elle fait intervenir la valeur, au centre de la bande critique, de la fonction L-symplectique correspondante.

1. Introduction

It has been almost 20 years since two of us proposed a rather speculative approach to the problem of restriction of irreducible representations from SO_n to SO_{n-1}

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[**24, 25**]. Our predictions depended on the Langlands parametrization of irreducible representations, using L-packets and L-parameters. Since then, there has been considerable progress in the construction of local L-packets, as well as on both local and global aspects of the restriction problem. We thought it was a good time to review the precise conjectures which remain open, and to present them in a more general form, involving restriction problems for all of the classical groups.

Let k be a local field equipped with an automorphism σ with $\sigma^2 = 1$ and let k_0 be the fixed field of σ. Let V be a vector space over k with a non-degenerate sesquilinear form and let $G(V)$ be the identity component of the classical subgroup of $\mathrm{GL}(V)$ over k_0 which preserves this form. There are four distinct cases, depending on whether the space V is orthogonal, symplectic, hermitian, or skew-hermitian. In each case, for certain non-degenerate subspaces W of V, we define a subgroup H of the locally compact group $G = G(V) \times G(W)$ containing the diagonally embedded subgroup $G(W)$, and a unitary representation ν of H. The local restriction problem is to determine

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}),$$

where π is an irreducible complex representation of G.

The basic cases are when $\dim V - \dim W = 1$ or 0, where ν is the trivial representation or a Weil representation respectively. When $\dim V - \dim W \geq 2$, this restriction problem is also known as the existence and uniqueness of Bessel or Fourier-Jacobi models in the literature. As in [**24**] and [**25**], our predictions involve the Langlands parametrization, in a form suggested by Vogan [**70**], and the signs of symplectic root numbers.

We show that the Langlands parameters for irreducible representations of classical groups (and for genuine representations of the metaplectic group) are complex representations of the Weil-Deligne group of k, of specified dimension and with certain duality properties. We describe these parameters and their centralizers in detail, before using their symplectic root numbers to construct certain distinguished characters of the component group. Our local conjecture states that there is a unique representation π in each generic Vogan L-packet, such that the dimension $d(\pi)$ is equal to 1. Furthermore, this representation corresponds to a distinguished character χ of the component group. For all other representations π in the L-packet, we predict that $d(\pi)$ is equal to 0. The precise statements are contained in Conjectures 17.1 and 17.3.

Although this material is largely conjectural, we prove a number of new results in number theory and representation theory along the way:

(i) In Proposition 5.2, we give a generalization of a formula of Deligne on orthogonal root numbers to the root numbers of conjugate orthogonal representations.
(ii) We describe the L-parameters of classical groups, and unitary groups in particular, in a much simpler way than currently exists in the literature; this is contained in Theorem 8.1.
(iii) We show in Theorem 11.1 that the irreducible representations of the metaplectic group can be classified in terms of the irreducible representations of odd special

orthogonal groups; this largely follows from fundamental results of Kudla-Rallis **[44]**, though the statement of the theorem did not appear explicitly in **[44]**.

(iv) We prove two theorems (cf. Theorems 15.1 and 16.1) that allow us to show the uniqueness of general Bessel and Fourier-Jacobi models over non-archimedean local fields. More precisely, we show that $d(\pi) \leq 1$ (cf. Corollaries 15.3, 16.2 and 16.3), reducing this to the basic cases when $\dim W^{\perp} = 0$ or 1, which were recently established by **[4]**, **[64]** and **[76]**. The same theorems allow us to reduce our local conjectures to these basic cases, as shown in Theorem 19.1.

One subtle point about our local conjecture is its apparent dependence on the choice of an additive character ψ of k_0 or k/k_0. Indeed, the choice of such a character ψ is potentially used in 3 places:

(a) the Langlands-Vogan parametrization (which depends on fixing a quasi-split pure inner form G_0 of G, a Borel subgroup B_0 of G_0, and a non-degenerate character on the unipotent radical of B_0);
(b) the definition of the distinguished character χ of the component group;
(c) the representation ν of H in the restriction problem.

Typically, two of the above depend on the choice of ψ, whereas the third one doesn't. More precisely, we have:

— in the orthogonal case, none of (a), (b) or (c) above depends on ψ; this explains why this subtlety does not occur in **[24]** and **[25]**.
— in the hermitian case, (a) and (b) depend on the choice of $\psi : k/k_0 \to \mathbb{S}^1$, but (c) doesn't.
— in the symplectic/metaplectic case, (a) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$, but (b) doesn't.
— in the odd skew-hermitian case, (b) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$, but (a) doesn't.
— in the even skew-hermitian case, (a) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$ but (b) doesn't.

Given this, we check in §18 that the dependence on ψ cancels out in each case, so that our local conjecture is internally consistent with respect to changing ψ. There is, however, a variant of our local conjectures which is less sensitive to the choice of ψ, but is slightly weaker. This variant is given in Conjecture 20.1. Finally, when all the data involved are unramified, we state a more refined conjecture; this is contained in Conjecture 21.3.

After these local considerations, we study the global restriction problem, for cuspidal tempered representations of adelic groups. Here our predictions involve the central values of automorphic L-functions, associated to a distinguished symplectic representation R of the L-group. More precisely, let $G = G(V) \times G(W)$ and assume that π is an irreducible cuspidal representation of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles of a global field F. If the vector space $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \bar{\nu}, \mathbb{C})$ is nonzero, our local conjecture implies that the global root number $\epsilon(\pi, R, \frac{1}{2})$ is equal to 1. If we assume π to be tempered, then our calculation of global root numbers and the general conjectures of Langlands and Arthur predict that π appears with multiplicity one in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$. We conjecture that the period integrals on the corresponding space of functions

$$f \mapsto \int_{H(k)\backslash H(\mathbb{A})} f(h) \cdot \overline{\nu(h)}\, dh$$

gives a nonzero element in $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \bar{\nu}, \mathbb{C})$ if and only if the central critical L-value $L(\pi, R, \frac{1}{2})$ is nonzero.

This first form of our global conjecture is given in §24, after which we examine the global restriction problem in the framework of Langlands-Arthur's conjecture on the automorphic discrete spectrum, and formulate a more refined global conjecture in §26. For this purpose, we formulate an extension of Langlands' multiplicity formula for metaplectic groups; see Conjecture 25.1.

One case in which all of these conjectures are known to be true is when $k = k_0 \times k_0$ is the split quadratic étale algebra over k_0, and V is a hermitian space over k of dimension n containing a codimension one nondegenerate subspace W. Then

$$G \cong \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0) \quad \text{and} \quad H \cong \mathrm{GL}_{n-1}(k_0).$$

Moreover, ν is the trivial representation. When k_0 is local, and π is a generic representation of $G = \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0)$, the local theory of Rankin-Selberg integrals [**34**], together with the multiplicity one theorems of [**4**], [**3**], [**66**], [**67**] and [**76**], shows that

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) = 1.$$

This agrees with our local conjecture, as the Vogan packets for $G = \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0)$ are singletons. If k_0 is global and π is a cuspidal representation of $G(\mathbb{A})$, then π appears with multiplicity one in the discrete spectrum. The global theory of Rankin-Selberg integrals [**34**] implies that the period integrals over $H(k)\backslash H(\mathbb{A})$ give a nonzero linear form on π if and only if

$$L(\pi, \mathrm{std}_n \otimes \mathrm{std}_{n-1}, 1/2) \neq 0,$$

where $L(\pi, \mathrm{std}_n \otimes \mathrm{std}_{n-1}, s)$ denotes the tensor product L-function. Again, this agrees with our global conjecture, since in this case, the local and global root numbers are all equal to 1, and

$$R = \mathrm{std}_n \otimes \mathrm{std}_{n-1} + \mathrm{std}_n^\vee \otimes \mathrm{std}_{n-1}^\vee.$$

In certain cases where the global root number $\epsilon = -1$, so that the central value is zero, we also make a prediction for the first derivative in §27. The cases we treat are certain orthogonal and hermitian cases, with $\dim W^\perp = 1$. We do not know if there is an analogous conjecture for the first derivative in the symplectic or skew-hermitian cases.

In a sequel to this paper, we will present some evidence for our conjectures, for groups of small rank and for certain discrete L-packets where one can calculate the distinguished character explicitly. We should mention that in a series of amazing papers [**77, 78, 74, 75**] and [**53**], Waldspurger and Mœglin-Waldspurger have established the local conjectures for special orthogonal groups, assuming some natural properties of the characters of representations in tempered L-packets. There is no doubt that their methods will extend to the case of unitary groups.

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2. Classical groups and restriction of representations

Let k be a field, not of characteristic 2. Let σ be an involution of k having k_0 as the fixed field. If $\sigma = 1$, then $k_0 = k$. If $\sigma \neq 1$, k is a quadratic extension of k_0 and σ is the nontrivial element in the Galois group $\mathrm{Gal}(k/k_0)$.

Let V be a finite dimensional vector space over k. Let

$$\langle -, - \rangle : V \times V \to k$$

be a non-degenerate, σ-sesquilinear form on V, which is ϵ-symmetric (for $\epsilon = \pm 1$ in $k^{\times}$):

$$\begin{aligned} \langle \alpha v + \beta w, u \rangle &= \alpha \langle v, u \rangle + \beta \langle w, u \rangle \\ \langle u, v \rangle &= \epsilon \cdot \langle v, u \rangle^{\sigma}. \end{aligned}$$

Let $G(V) \subset \mathrm{GL}(V)$ be the algebraic subgroup of elements T in $\mathrm{GL}(V)$ which preserve the form $\langle -, - \rangle$:

$$\langle Tv, Tw \rangle = \langle v, w \rangle.$$

Then $G(V)$ is a classical group, defined over the field k_0. The different possibilities for $G(V)$ are given in the following table.

(k, ϵ)	$k = k_0$, $\epsilon = 1$	$k = k_0$, $\epsilon = -1$	k/k_0 quadratic, $\epsilon = \pm 1$
$G(V)$	orthogonal group $\mathrm{O}(V)$	symplectic group $\mathrm{Sp}(V)$	unitary group $\mathrm{U}(V)$

In our formulation, a classical group will always be associated to a space V, so the hermitian and skew-hermitian cases are distinct. Moreover, the group $G(V)$ is connected except in the orthogonal case. In that case, we let $\mathrm{SO}(V)$ denote the connected component, which consists of elements T of determinant $+1$, and shall refer to $\mathrm{SO}(V)$ as a connected classical group. We will only work with connected classical groups in this paper.

If one takes k to be the quadratic algebra $k_0 \times k_0$ with involution $\sigma(x, y) = (y, x)$ and V a free k-module, then a non-degenerate form $\langle -, - \rangle$ identifies the $k = k_0 \times k_0$ module V with the sum $V_0 + V_0^{\vee}$, where V_0 is a finite dimensional vector space over k_0 and $V_0^{\vee}$ is its dual. In this case $G(V)$ is isomorphic to the general linear group $\mathrm{GL}(V_0)$ over k_0.

If G is a connected, reductive group over k_0, the pure inner forms of G are the groups G' over k_0 which are obtained by inner twisting by elements in the pointed set $H^1(k_0, G)$. If $\{g_\sigma\}$ is a one cocycle on the Galois group of the separable closure k_0^s with values in $G(k_0^s)$, the corresponding pure inner form G' has points

$$G'(k_0) = \{a \in G(k_0^s) : a^\sigma = g_\sigma a g_\sigma^{-1}\}.$$

The group G' is well-defined up to inner automorphism over k_0 by the cohomology class of g_σ, so one can speak of a representation of $G'(k_0)$.

For connected, classical groups $G(V) \subset \mathrm{GL}(V)$, the pointed set $H^1(k_0, G)$ and the pure inner forms G' correspond bijectively to forms V' of the space V with its sesquilinear form $\langle , \rangle$ (cf. [**40**, § 29D and § 29E]).

Lemma 2.1. — 1. *If $G = \mathrm{GL}(V)$ or $G = \mathrm{Sp}(V)$, then the pointed set $H^1(k_0, G) = 1$ and there are no nontrivial pure inner forms of G.*
2. *If $G = \mathrm{U}(V)$, then elements of the pointed set $H^1(k_0, G)$ correspond bijectively to the isomorphism classes of hermitian (or skew-hermitian) spaces V' over k with $\dim(V') = \dim(V)$. The corresponding pure inner form G' of G is the unitary group $\mathrm{U}(V')$.*
3. *If $G = \mathrm{SO}(V)$, then elements of the pointed set $H^1(k_0, G)$ correspond bijectively to the isomorphism classes of orthogonal spaces V' over k with $\dim(V') = \dim(V)$ and $\mathrm{disc}(V') = \mathrm{disc}(V)$. The corresponding pure inner form G' of G is the special orthogonal group $\mathrm{SO}(V')$.*

Now let $W \subset V$ be a subspace, which is non-degenerate for the form $\langle -, - \rangle$. Then $V = W + W^\perp$. We assume that

1) $\epsilon \cdot (-1)^{\dim W^\perp} = -1$
2) $W^\perp$ is a split space.

When $\epsilon = -1$, so dim $W^\perp = 2n$ is even, condition 2) means that $W^\perp$ contains an isotropic subspace X of dimension n. It follows that $W^\perp$ is a direct sum

$$W^\perp = X + Y,$$

with X and Y isotropic. The pairing $\langle -, - \rangle$ induces a natural map

$$Y \longrightarrow \mathrm{Hom}_k(X, k) = X^\vee$$

which is a k_0-linear isomorphism (and k-anti-linear if $k \neq k_0$). When $\epsilon = +1$, so dim $W^\perp = 2n + 1$ is odd, condition 2) means that $W^\perp$ contains an isotropic subspace X of dimension n. It follows that

$$W^\perp = X + Y + E,$$

where E is a non-isotropic line orthogonal to $X + Y$, and X and Y are isotropic. As above, one has a k_0-linear isomorphism $Y \cong X^\vee$.

Let $G(W)$ be the subgroup of $G(V)$ which acts trivially on $W^\perp$. This is the classical group, of the same type as $G(V)$, associated to the space W. Choose an $X \subset W^\perp$ as above, and let P be the parabolic subgroup of $G(V)$ which stabilizes a complete flag of (isotropic) subspaces in X. Then $G(W)$, which acts trivially on both X and $X^\vee$, is contained in a Levi subgroup of P, and acts by conjugation on the unipotent radical N of P.

The semi-direct product $H = N \rtimes G(W)$ embeds as a subgroup of the product group $G = G(V) \times G(W)$ as follows. We use the defining inclusion $H \subset P \subset G(V)$ on the first factor, and the projection $H \to H/N = G(W)$ on the second factor. When $\epsilon = +1$, the dimension of H is equal to the dimension of the complete flag variety of G. When $\epsilon = -1$, the dimension of H is equal to the sum of the dimension of the complete flag variety of G and half of the dimension of the vector space W over k_0.

We call a pure inner form $G' = G(V') \times G(W')$ of the group G *relevant* if the space W' embeds as a non-degenerate subspace of V', with orthogonal complement isomorphic to $W^\perp$. We note:

Lemma 2.2. — *Suppose k is non-archimedean.*

(i) *In the orthogonal and hermitian cases, there are 4 pure inner forms of $G = G(V) \times G(W)$ and among these, exactly two are relevant. Moreover, among the two relevant pure inner forms, exactly one is a quasi-split group.*

(ii) *In the symplectic case, there is exactly one pure inner form of $G = G(V) \times G(W)$, which is necessarily relevant.*

(iii) *In the skew-hermitian case, there are 4 pure inner forms of $G = G(V) \times G(W)$, exactly two of which are relevant. When* $\dim V$ *is odd, the two relevant pure inner forms are both quasi-split, and when* $\dim V$ *is even, exactly one of them is quasi-split.*

Proof. — The statement (i) follows from the fact that an odd dimensional split quadratic space is determined by its discriminant and that there is a unique split hermitian space of a given even dimension. The statements (ii) and (iii) are similarly treated. □

Given a *relevant* pure inner form $G' = G(V') \times G(W')$ of G, one may define a subgroup $H' \subset G'$ as above. In this paper, we will study the restriction of irreducible complex representations of the groups $G' = G(V') \times G(W')$ to the subgroups H', when k is a local or a global field.

3. Selfdual and conjugate-dual representations

Let k be a local field, and let k^s be a separable closure of k. In this section, we will define selfdual and conjugate-dual representations of the Weil-Deligne group $WD(k)$ of k.

When $k = \mathbb{R}$ or $\mathbb{C}$, we define $WD(k)$ as the Weil group $W(k)$ of k, which is an extension of $\mathrm{Gal}(k^s/k)$ by $\mathbb{C}^\times$, and has abelianization isomorphic to $k^\times$. A representation of $WD(k)$ is, by definition, a completely reducible (or semisimple) continuous homomorphism

$$\varphi : WD(k) \to \mathrm{GL}(M),$$

where M is a finite dimensional complex vector space. When k is non-archimedean, the Weil group $W(k)$ is the dense subgroup $I \rtimes F^{\mathbb{Z}}$ of $\mathrm{Gal}(k^s/k)$, where I is the inertia group and F is a geometric Frobenius. We normalize the isomorphism

$$W(k)^{ab} \to k^{\times}$$

of local class field theory as in Deligne [**13, 14, 15**], taking F to a uniformizing element of $k^{\times}$. This defines the norm character

$$|-| : W(k) \to \mathbb{R}^{\times}, \quad \text{with } |F| = q^{-1}.$$

We define $WD(k)$ as the product of $W(k)$ with the group $\mathrm{SL}_2(\mathbb{C})$. A representation is a homomorphism

$$\varphi : WD(k) \to \mathrm{GL}(M)$$

with

(i) φ trivial on an open subgroup of I,
(ii) $\varphi(F)$ semi-simple,
(iii) $\varphi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(M)$ algebraic.

The equivalence of this formulation of representations with that of Deligne [**13, 14, 15**], in which a representation is a homomorphism $\rho : W(k) \to \mathrm{GL}(M)$ and a nilpotent endomorphism N of M which satisfies $Ad\rho(w)(N) = |w| \cdot N$, is given in [**26**, § 2, Proposition 2.2].

We say two representations M and M' of $WD(k)$ are isomorphic if there is a linear isomorphism $f : M \to M'$ which commutes with the action of $WD(k)$. If M and M' are two representations of $WD(k)$, we have the direct sum representation $M \oplus M'$ and the tensor product representation $M \otimes M'$. The dual representation $M^{\vee}$ is defined by the natural action on $\mathrm{Hom}(M, \mathbb{C})$, and the determinant representation $\det(M)$ is defined by the action on the top exterior power. Since $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^{\times}$ is abelian, the representation $\det(M)$ factors through the quotient $W(k)^{ab} \to k^{\times}$ of $WD(k)$.

We now define certain selfdual representations of $WD(k)$. We say the representation M is orthogonal if there is a non-degenerate bilinear form

$$B : M \times M \to \mathbb{C}$$

which satisfies

$$\begin{cases} B(\tau m, \tau n) = B(m, n) \\ B(n, m) = B(m, n), \end{cases}$$

for all τ in $WD(k)$.

We say M is symplectic if there is a non-degenerate bilinear form B on M which satisfies

$$\begin{cases} B(\tau m, \tau n) = B(m, n) \\ B(n, m) = -B(m, n), \end{cases}$$

for all τ in $WD(k)$.

In both cases, the form B gives an isomorphism of representations

$$f : M \to M^{\vee},$$

whose dual
$$f^\vee : M = M^{\vee\vee} \to M^\vee$$
satisfies
$$f^\vee = b \cdot f, \quad \text{with } b = \text{the sign of } B.$$

We now note:

Lemma 3.1. — *Given any two non-degenerate forms B and B' on M preserved by $WD(k)$ with the same sign $b = \pm 1$, there is an automorphism T of M which commutes with $WD(k)$ and such that $B'(m,n) = B(Tm, Tn)$.*

Proof. — Since M is semisimple as a representation of $WD(k)$, we may write
$$M = \bigoplus_i V_i \otimes M_i$$
as a direct sum of irreducible representations with multiplicity spaces V_i. Each M_i is either selfdual or else $M_i^\vee \cong M_j$ for some $i \neq j$, in which case $\dim V_i = \dim V_j$. So we may write
$$M = \left(\bigoplus_i V_i \otimes M_i\right) \oplus \left(\bigoplus_j V_j \otimes (P_j + P_j^\vee)\right)$$
with M_i irreducible selfdual and P_j irreducible but $P_j \ncong P_j^\vee$. Since any non-degenerate form B remains non-degenerate on each summand above, we are reduced to the cases:

(a) $M = V \otimes N$ with N irreducible and selfdual, in which case the centralizer of the action of $WD(k)$ is $\mathrm{GL}(V)$;

(b) $M = (V \otimes P) \oplus (V \otimes P^\vee)$, with P irreducible and $P \ncong P^\vee$, in which case the centralizer of the action of $WD(k)$ is $\mathrm{GL}(V) \times \mathrm{GL}(V)$.

In case (a), since N is irreducible and selfdual, there is a unique (up to scaling) $WD(k)$-invariant non-degenerate bilinear form on N; such a form on N has some sign b_N. Thus, giving a $WD(k)$-invariant non-degenerate bilinear form B on M of sign b is equivalent to giving a non-degenerate bilinear form on V of sign $b \cdot b_N$. But it is well-known that any two non-degenerate bilinear forms of a given sign are conjugate under $\mathrm{GL}(V)$. This takes care of (a).

In case (b), the subspaces $V \otimes P$ and $V \otimes P^\vee$ are necessarily totally isotropic. Moreover, there is a unique (up to scaling) $WD(k)$-invariant pairing on $P \times P^\vee$. Thus to give a $WD(k)$-invariant non-degenerate bilinear form B on M of sign b is equivalent to giving a non-degenerate bilinear form on V. But any two such forms are conjugate under the action of $\mathrm{GL}(V) \times \mathrm{GL}(V)$ on $V \times V$. This takes care of (b) and the lemma is proved. $\square$

When M is symplectic, $\dim(M)$ is even and $\det(M) = 1$. When M is orthogonal, $\det(M)$ is an orthogonal representation of dimension 1. These representations correspond to the quadratic characters
$$\chi : k^\times \to \langle \pm 1 \rangle.$$

Since $\mathrm{char}(k) \neq 2$, the Hilbert symbol gives a perfect pairing

$$(-,-) : k^\times/k^{\times 2} \times k^\times/k^{\times 2} \to \langle \pm 1 \rangle.$$

We let $\mathbb{C}(d)$ be the one dimensional orthogonal representation given by the character $\chi_d(c) = (c, d)$.

We also note the following elementary result:

Lemma 3.2. — *If M_i is selfdual with sign b_i, for $i = 1$ or 2, then $M_1 \otimes M_2$ is selfdual with sign $b_1 \cdot b_2$.*

Proof. — If M_i is selfdual with respect to a form B_i of sign b_i, then $M_1 \otimes M_2$ is selfdual with respect to the tensor product $B_1 \otimes B_2$ which has sign $b_1 \cdot b_2$. □

Next, assume that σ is a nontrivial involution of k, with fixed field k_0. Let s be an element of $W(k_0)$ which generates the quotient group

$$W(k_0)/W(k) = \mathrm{Gal}(k/k_0) = \langle 1, \sigma \rangle.$$

If M is a representation of $WD(k)$, let M^s denote the conjugate representation, with the same action of $\mathrm{SL}_2(\mathbb{C})$ and the action $\tau_s(m) = s\tau s^{-1}(m)$ for τ in $W(k)$.

We say the representation M is conjugate-orthogonal if there is a non-degenerate bilinear form $B : M \times M \to \mathbb{C}$ which satisfies

$$\begin{cases} B(\tau m, s\tau s^{-1} n) = B(m, n) \\ B(n, m) = B(m, s^2 n), \end{cases}$$

for all τ in $WD(k)$. We say M is conjugate-symplectic if there is a non-degenerate bilinear form on M which satisfies

$$\begin{cases} B(\tau m, s\tau s^{-1} n) = B(m, n) \\ B(n, m) = -B(m, s^2 n), \end{cases}$$

for all τ in $WD(k)$. In both cases, the form B gives an isomorphism of representations

$$f : M^s \to M^\vee,$$

whose conjugate-dual

$$(f^\vee)^s : M^s \longrightarrow ((M^s)^\vee)^s \xrightarrow{\varphi(s^2)} M^\vee$$

satisfies

$$(f^\vee)^s = b \cdot f \quad \text{with } b = \text{the sign of } B.$$

We now note:

Lemma 3.3. — *Given two such non-degenerate forms B and B' on M with the same sign and preserved by $WD(k)$, there is an automorphism of M which commutes with $WD(k)$ and such that $B'(m, n) = B(Tm, Tn)$.*

Proof. — The proof is similar to that of Lemma 3.1. As before, we may reduce to the following two cases: