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COMPUTING WITH THE LAMBDA ALGEBRA

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We are trying to compute the homotopy groups of spheres. This is an old problem now, and a very deep one; and a lecture on the subject is likely to be very technical. A number of the experts who know these technicalities are among the participants in this Congress. If I were to give a technical talk on my work, a few of you would already know what I was going to say before I had begun, but most of you would still not know what I had said after I had finished. Moreover, I have published elsewhere [6] a detailed account of several aspects of the problem that I am currently engaged in.

Accordingly, I would like to confine myself here to some remarks that I hope will be appreciated by everybody. On the one hand, I would like to give an idea of how we have managed to convert an effectively computable but realistically intractable problem into a tractable and really computable one. Here I will oversimplify the description, since the interested reader can refer to more detailed versions in the literature.

On the other hand, I would like to share with you some reflections on the meaning of "proof" as it is variously used in our various disciplines. When is a proof really a proof? Let me begin with an assertion made some years ago by the American humorist Al Capp, or rather by a character in his comic strip Li'l Abner.

Mammy Yokum's Principle : Good is better than evil, because it's nicer.

Mammy Yokum's method of proof is well known in many other fields of human endeavor, but I submit that it has been neglected by mathematicians and computer scientists

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There is a saying among mathematicians that only a graduate student really knows what a proof is. Perhaps it is necessary to have published an erroneous result in order to appreciate this.

When we think we have a proof, we submit it to three tests. First, we try to find a mistake in it. Second, we submit it for publication, and the referee tries to find a mistake in it. Third, it is published, and everyone else tries to find a mistake in it.

This leads me to what I think of as the Mammy Yokum Test for a proof in mathematics : a proof of a mathematical result is a good proof if nobody has found a mistake in it.

Later we will offer a Mammy Yokum Test for the correctness of a computation.

1. ALGEBRAIC APPROACHES TO THE HOMOTOPY PROBLEM.

Homotopy groups are algebraic objects occurring in, and defined in, a purely topological setting. The definition is a matter of topological spaces and continuous functions. However, a dozen years after the definition had been codified, the great difficulty of the problem of computing these groups had become apparent; Hopf complained around 1950 that almost every known result had been obtained by a different method [4].

Great progress was made in the 1950s and 1960s, but the improvement came at the expense of elaborate techniques, and the result was a bewildering confusion of data, in which a variety of patterns can be seen to interact in complicated and often mysterious ways. It is known, for example, that every possible positive integer occurs as the order of an element in the homotopy groups of spheres. I attach a little table of the homotopy groups of the 6-sphere, extracted from Toda's 1962 book [7], and invite you to try to extrapolate to the next few groups.

The table gives, for each n, the order $O(n)$ of the homotopy group $\pi_n(S^6)$. A zero denotes a trivial group, and ∞ denotes an infinite cyclic group.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
O(n)	0	0	0	0	0	∞	2	2	24	0	∞	2	60	48	8

n	16	17	18	19	20	21	22	23	24	25
O(n)	144	2016	240	6	24	360	2016	16	288	8448

More optimistically, Brown proved in 1957 that the homotopy groups of any finite complex are effectively computable [2]. However, Brown himself emphasized that his algorithms were not intended to be of any practical use.

Meanwhile, work of Steenrod, Cartan, Serre, and Adams led to the consideration of certain algebraic approximations to the homotopy problem, which are more amenable to computation. In particular the Adams spectral sequence converges to a filtered version of homotopy groups, and its E_2 term is an algebraic object for which any finite range can be obtained from a variety of algorithmic processes. The issue becomes one of efficiency : all algorithms are effective, but some are more effective than others.

We will confine ourselves now to the problem of computing the E_2 term of the Adams spectral sequence for spheres, for $p = 2$. At least four different methods have been used. Since the E_2 is the cohomology of the Steenrod algebra, it can be obtained as the homology of the cobar construction; but this construction is very large and the method is too slow and cumbersome. Adams used a minimal resolution, but this method seems awkward for large computations, although recently Bruner has had surprising success with it on a computer. The May spectral sequence is well suited to hand computation but to date has not been carried any further on a machine.

We focus on a fourth method, which obtains the Adams E_2 term as the homology of the lambda algebra. When this algebra was announced in the 1960s, it did not seem very promising for computation, but Ed Curtis showed the way. George Whitehead has done some extensive calculations by hand, and the ideas of Curtis are amenable to algorithmic development and to machine computation.

2. THE LAMBDA ALGEBRA.

For each prime p there is a lambda algebra. To simplify the discussion we will only consider $p = 2$. In this case Λ is an associative bi-graded differential algebra over the field of 2 elements, with a generator λ_n in each non-negative dimension n. The algebra is *not* commutative, so your monomials are ordered products λ_I where $I = (i_1, i_2, \ldots, i_s)$ is a sequence of non-negative integers. It is natural to write I as an abbreviated notation for λ_I. The algebra is defined by the relations

$$\lambda_i \lambda_{2i+n+1} = \sum_{j \geq 0} A(n,j) \lambda_{i+n-j} \lambda_{2i+1+j} \qquad (i \geq 0, n \geq 0)$$

and the differential

$$d(\lambda_{n-1}) = \sum_{j \geq 1} A(n,j) \lambda_{n-j-1} \lambda_{j-1} \qquad (n \geq 1)$$

where $A(n,j)$ denotes the binomial coefficient $\binom{n-j-1}{j}$ reduced $\bmod 2$. The bi-grading of a monomial indexed by I may be written (r, s) where s, as above, is the length of I, and $r = i_1 + \ldots + i_s$. Using the relations we can express all monomials in terms of the "admissible" ones satisfying $2i_j \geq i_{j+1} \quad (1 \leq j \leq s-1)$.

Because of the non-commutativity, the algebra grows very fast. Just by counting the elements (using a computer, of course) we find an exponential growth rate of 1.79 with respect to the r grading. This was recently explained by Flajolet and Prodinger [3]. Since 1.79^{10} is about 345, we see that if you have an algorithm that is linear with respect to the number of elements in the admissible basis, and if you can compute E_2 from dimension $r = 30$ to dimension $r = 40$ in a month, then you can go from 40 to 50 in about thirty years. This may be effective, but it is not effective enough.

As anyone knows who has worked in computational linear algebra, the key is to choose the right basis. Curtis's method of choosing bases may have been motivated topologically, but it has the interesting effect of allowing us to set aside the vast majority of monomials as being irrelevant. The following discussion is intended to give a rough idea of what I mean by this, and to identify the properties of the lambda algebra that make this possible.