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LEMPERT MAPPINGS AND HOLOMORPHIC MOTIONS IN $\mathbf{C}^n$

by

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Abstract. — The purpose of this note is twofold: to discuss the concept of holomorphic motions and phenomena of Mañé-Sad-Sullivan type in several complex variables and secondly, to compare the different notions of Beltrami differentials in CR-geometry which have appeared in [4] and [7].

1. Introduction

Holomorphic motions in the complex plane $\mathbf{C}$ are isotopies of subsets $A \subset \mathbf{C}$ for which the dependence on the "time" parameter is holomorphic. This simple notion has been important in explaining a number of different questions in complex analysis, in particular the rigidity phenomena in complex dynamics and the role of quasiconformal mappings in holomorphic deformations.

It was Mañé, Sad and Sullivan **[9]** who first realized that for time-holomorphic isotopies one can forget all smoothness requirements in space variables and thus produce almost automatic rigidity results in various contexts. Given a subset $A \subset \overline{\mathbf{C}}$, it is simply enough to define a holomorphic motion of A as a mapping $f : \Delta \times A \to \overline{\mathbf{C}}$, where $\Delta = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$, such that

(i) for any fixed $a \in A$, the map $\lambda \to f(\lambda, a)$ is holomorphic in Δ
(ii) for any fixed $\lambda \in \Delta$, the map $a \to f(\lambda, a) = f_\lambda(a)$ is an injection and
(iii) the mapping f_0 is the identity on A.

Then f is automatically continuous in $\overline{A} \times \overline{\mathbf{C}}$ and the restrictions $f_\lambda(.)$ are quasisymmetric mappings **[9]**; in case $A = \overline{\mathbf{C}}$ they are quasiconformal with the precise

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bound on the dilatation

$$K(f_\lambda) \le \frac{1+|\lambda|}{1-|\lambda|}. \tag{1}$$

The picture was then completed by Slodkowski **[12]** who proved the so called Generalized Λ-lemma, that a holomorphic motion of any set $A \subset \overline{\mathbf{C}}$ extends to a motion of the whole $\overline{\mathbf{C}}$.

In this setting it is natural to look for similar phenomena in several complex variables, when the sets moving are of higher dimension. However, one quickly sees that the simple minded generalization does not work: Let, for example, $S(x) = x/|x|$ for $x \in \mathbf{R} \setminus \{0\}$, $S(0) = 0$ and define

$$f_\lambda(z,w) = (z + \lambda S(\mathrm{Re}\{w\}), w).$$

Then f_λ is holomorphic in λ and injective but not even continuous in $\mathbf{C}^2$.

Our first goal in this note is to introduce the proper notion or point of view to holomorphic motions in several complex variables and then show the existence of the first nontrivial examples, results of Mañé-Sad-Sullivan type. We expect that similar phenomena occur, in fact, in much larger setups.

Remark. — The generalizations to the case where the *parameter* space is higher dimensional were studied by Adrien Douady in his work **[3]**.

If there are to be holomorphic motions in $\mathbf{C}^n$, the one-dimensional theory suggests that they are connected to a notion of quasiconformality. Therefore recall that in several complex variables the appropriate concepts are the quasiconformal mappings on CR-structures **[4]**, or mappings on boundaries of pseudoconvex domains which firstly are contact transforms, *i.e.* preserve the horizontal (complex) lines of the tangent spaces

$$H_p\partial D = T_p\partial D \cap JT_p\partial D$$

where J is the complex structure as a mapping of $T_p\mathbf{C}^2$, and secondly, are there quasiconformal with respect to the corresponding Levi Form, *i.e.*

$$K(p) = \frac{\sup\{L(F_*X, F_*X) : X \in H_p\partial D,\ L(X,X) = 1\}}{\inf\{L(F_*X, F_*X) : X \in H_p\partial D,\ L(X,X) = 1\}} \le K \tag{2}$$

for all $p \in \partial D$.

The same direction is, actually, suggested also by the approach of Slodkowski **[12]**. He viewed holomorphic motions (or their graphs) as disjoint analytic disks in $\mathbf{C}^2$. Namely given such a motion $f : \Delta \times A \to \mathbf{C}$ each point $a \in A$ defines a holomorphic disk $D_a \subset \mathbf{C}^2$, a holomorphic image of Δ, by

$$D_a = \{(\lambda, f(\lambda, a)) : \lambda \in \Delta\} \tag{3}$$

and these disks are clearly pointwise disjoint. Conversely, given a family of analytic disks of the form (3) with $D_a \cap D_b = \varnothing$ when $a \neq b$, they define a holomorpic motion $\Psi(\lambda, f(0,a)) = f(\lambda, a)$. (The extension of a given motion was then obtained by studying certain totally real tori whose polynomial hulls were shown to consists of disjoint families of suitable analytic disks.)

Interpreting the Mañé-Sad-Sullivan result in the language (3) of disjoint analytic disks, for a motion of the whole complex plane the disks D_a, $a \in \mathbf{C}$, fill in the domain $\Delta \times \mathbf{C}$. And when we move along the disks with λ, in the transverse direction *i.e.* on the *complex* lines of the corresponding tangent spaces of $\partial\Delta(|\lambda|) \times \mathbf{C}$ the mappings $(0,a) \mapsto (\lambda, f(\lambda, a))$ are now quasiconformal by the original λ-lemma.

This picture makes it very suggestive that similar phenomena should occur in other situations in $\mathbf{C}^2$ or $\mathbf{C}^n$ as well. That is, for suitable families of analytic disks one should expect that moving holomorphically along these disks yields automatic quasiconformality in transverse horizontal directions, quasiconformality in the Koranyi-Reimann sense, with the bound (1) on the dilatation. The philosophy of holomorphic motions in $\mathbf{C}^n$ would then be not that there is one strict definition of these motions but rather that there are several natural situations that share the common features described here.

To show that there do exist nontrivial holomorphic motions in the above sense in $\mathbf{C}^2$ (the choice $n = 2$ is made for simplicity) we make use of the theory developed by Lempert **[5]**-**[8]** and consider bounded strictly $\mathbf{R}$-convex smooth subdomains $D \subset \mathbf{C}^2$ and their generalizations the strictly linearly convex domains. The latter class consists of smooth bounded domains with the property that for each boundary point $p \in \partial D$ the horizontal space $H_p\partial D$ does not intersect $\bar{D}\backslash\{p\}$ and that $H_p\partial D$ has precisely first order contact with ∂D at p. That is, there exists $c > 0$ such that

$$\operatorname{dist}(q, H_p\partial D) \geq c \cdot \operatorname{dist}(p,q)^2, \qquad q \in D.$$

In particular, strictly convex domains are strictly linearly convex which in turn are strictly pseudoconvex.

As shown by Lempert in strictly linearly convex domains extremal Kobayashi disks are especially well behaved. For this recall that in any bounded domain containing the origin the Kobayashi indicatrix $\bar{I}$ of D is defined by

$$\bar{I} = \{f'(0) : \ f : \Delta \to D \text{ is holomorphic and } f(0) = 0\}.$$

If $v \in \partial\bar{I}$, a holomorphic mapping $f = f_v : \Delta \to D$ such that $f(0) = 0$ and $f'(0) = v$ is then called an extremal map corresponding to the vector v. In strictly linearly convex domains extremal disks are uniquely determined by v, a fact no longer true for general pseudoconvex domains. This enables us to simply define

$$\Psi : \Delta \times \partial\bar{I} \to \mathbf{C}^2, \qquad \Psi(\lambda, v) = \frac{f_v(\lambda)}{\lambda}$$

and we can describe a full counterpart of the Mañé-Sad-Sullivan result, a holomorphic motion of $\partial\bar{I}$.

Theorem 1. — *Let D be a strictly linearly convex domain containing the origin and $\Psi : \Delta \times \partial \bar{I} \to \mathbf{C}^2$ be defined by $\Psi(\lambda, v) = \lambda^{-1} f_v(\lambda)$. Then Ψ satisfies the following properties:*

(1) $\Psi(0, \cdot) = \mathrm{Id}\,|_{\partial \bar{I}}$;
(2) $\Psi(\cdot, v) : \Delta \to \mathbf{C}^2$ *is holomorphic;*
(3) $\Psi(\lambda, \cdot) : \partial \bar{I} \to A_\lambda$ *is a contact mapping where* $A_\lambda = \Psi(\lambda, \partial \bar{I})$ *is the boundary of a strictly pseudoconvex domain. In particular,* Ψ *is continuous in* $\Delta \times \partial \bar{I}$;
(4) $\Psi(\lambda, \cdot) : \partial \bar{I} \to A_\lambda$ *is* $K(\lambda)$*-quasiconformal with* $K(\lambda) \leq \frac{1+|\lambda|}{1-|\lambda|}$, $\lambda \in \Delta$.

We should mention that statement (4) is the new result proven here; statements (1), (2) and (3), due to Lempert, are being included for the sake of completeness. To obtain the optimal dilatation bound we turn to our second goal, to compare the different notions of Beltrami differentials in contact geometry and CR-manifolds, introduced respectively, by Koranyi and Reimann **[4]** and Lempert **[7]**. This with required preliminary material will be presented in the next section.

2. Inner actions and Beltrami differentials

It will be convenient start with a version of the Riemann mapping theorem in $\mathbf{C}^n$ due to Lempert (**[5]**, **[6]**), and use the formalism introduced by Semmes **[11]**. These Riemann mappings preserve the complex structure to some extent but are flexible enough to yield general existence results. In more precise terms, Lempert considered mappings $\rho : \bar{B} \to \bar{D}$ from the unit ball in $\mathbf{C}^n$ onto domains $D \subset \mathbf{C}^n$ containing the origin which satisfy the following three requirements:

(1) $\rho : \bar{B} \backslash \{0\} \to \bar{D} \backslash \{0\}$ is a smooth diffeomorphism and $\rho : \bar{B} \to \bar{D}$ is bilipshitz;
(2) ρ restricted to any complex line through the origin is holomorphic;
(3) ρ restricted to the boundary of any ball B_r centered at the origin and of radius $0 < r \leq 1$ is contact, *i.e.*

$$\rho_* H\partial B_r = H\partial D_r \qquad\qquad (D_r = \rho B_r).$$

For the last condition recall that when a domain is strictly pseudoconvex the horizontal tangent bundle $H\partial D = T\partial D \cap JT\partial D$, where J is the complex structure, defines a contact structure on the boundary.

In what follows a mapping with the above properties (1)-(3) will be called a Lempert mapping. The basic existence result is then:

***Theorem A* (Lempert)**. — *Let D be a strictly linearly convex domain. Then there exists a Lempert mapping $\rho : \bar{B} \to \bar{D}$.*

A very nice exposition of the properties of the Lempert mappings was given by Semmes in **[11]**. The statement and proof of Theorem A, for example, may be found in **[11]** in the case of strictly convex domains and it is based essentially on the results