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CAYLEY TRANSFORM AND GENERALIZED WHITTAKER MODELS FOR IRREDUCIBLE HIGHEST WEIGHT MODULES

by

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Dedicated to Professor Ryoshi Hotta on his sixtieth birthday

Abstract. — We study the generalized Whittaker models for irreducible admissible highest weight modules $L(\tau)$ for a connected simple Lie group G of Hermitian type, by using certain invariant differential operators $\mathcal{D}_{\tau^*}$ of gradient type on the Hermitian symmetric space $K\backslash G$. It is shown that each $L(\tau)$ embeds, with nonzero and finite multiplicity, into the generalized Gelfand-Graev representation $\Gamma_{m(\tau)}$ attached to the unique open orbit $\mathcal{O}_{m(\tau)}$ (through the Kostant-Sekiguchi correspondence) in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. The embeddings can be intrinsically analyzed by means of the Cayley transform which carries the bounded realization of $K\backslash G$ to unbounded one. If $L(\tau)$ is unitarizable, the space $\mathcal{Y}(\tau)$ of infinitesimal homomorphisms from $L(\tau)$ into $\Gamma_{m(\tau)}$ can be described in terms of the principal symbol at the origin of the differential operator $\mathcal{D}_{\tau^*}$. For the classical groups $G = SU(p,q)$, $Sp(n,\mathbb{R})$ and $SO^*(2n)$, the space $\mathcal{Y}(\tau)$ is clearly understood through the oscillator representations of reductive dual pairs.

Résumé **(Transformation de Cayley et modèles de Whittaker généralisés pour les modules irréductibles de plus haut poids)**

Soit G un groupe de Lie connexe simple de type hermitien. On considère les G-modules irréductibles admissibles $L(\tau)$ de plus haut poids. Dans cet article, nous étudions les modèles de Whittaker généralisés pour $L(\tau)$ en utilisant certains opérateurs différentiels de type gradient $\mathcal{D}_{\tau^*}$ sur l'espace hermitien symétrique $K\backslash G$. Il est montré que chaque $L(\tau)$ apparaît, avec une multiplicité finie et non nulle, dans la représentation de Gelfand-Graev généralisée $\Gamma_{m(\tau)}$ qui est attachée à l'unique orbite ouverte $\mathcal{O}_{m(\tau)}$ (par la correspondance de Kostant-Sekiguchi) dans la variété $\mathcal{V}(L(\tau))$ associée à $L(\tau)$. On peut analyser intrinsèquement les isomorphismes de $L(\tau)$ dans $\Gamma_{m(\tau)}$ au moyen de la transformation de Cayley qui donne un rapport entre la réalisation de $K\backslash G$ comme domaine borné et celle comme domaine non borné. Si $L(\tau)$ est unitarisable, l'espace $\mathcal{Y}(\tau)$ des homomorphismes infinitésimaux de $L(\tau)$ dans $\Gamma_{m(\tau)}$ s'exprime par le symbole principal à l'origine de l'opérateur différentiel $\mathcal{D}_{\tau^*}$. Pour les groupes classiques $G = SU(p,q)$, $Sp(n,\mathbb{R})$ et $SO^*(2n)$, on peut comprendre l'espace $\mathcal{Y}(\tau)$ en utilisant les représentations oscillateur pour les paires duales réductives.

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Introduction

For a semisimple algebraic group G, the generalized Gelfand-Graev representations introduced by Kawanaka **[14]** form a family of representations of G induced from certain one-dimensional characters of various unipotent subgroups. By construction, each of these induced G-modules is naturally attached to a nilpotent G-orbit $\mathcal{O}_G$ in the Lie algebra through the Dynkin-Kostant theory. The original (non generalized) Gelfand-Graev representations are induced from nondegenerate characters of a maximal unipotent subgroup, and they correspond to the principal nilpotent orbits. We say that an irreducible representation π of G has a generalized Whittaker model of type $\mathcal{O}_G$ if π admits an embedding into the generalized Gelfand-Graev representation attached to $\mathcal{O}_G$. The problem of describing the generalized Whittaker models is important not only in representation theory but also in connection with the theory of automorphic forms.

Generalized Whittaker models (or vectors) for irreducible representations of G have been studied by many authors (e.g., **[14]**, **[15]**, **[26]**, **[22]**, **[24]**, **[39]**, etc.). For real or complex groups, it is Kostant **[18]** who initiated a systematic study on the existence of nonzero Whittaker vectors attached to the principal nilpotent orbits of quasi-split groups, in connection with the primitive ideals of the irreducible representations in question. Later, some results of Kostant have been extended by Matumoto to those on generalized Whittaker vectors associated to arbitrary (not necessarily principal) nilpotent orbits $\mathcal{O}_G$. In fact, it is shown in **[22]** that the Harish-Chandra module of an irreducible admissible representation π has a nonzero generalized Whittaker vector of type $\mathcal{O}_G$ only if the nilpotent orbit $\mathcal{O}_G$ is contained in the associated variety of the primitive ideal $\operatorname{Ann}\pi$ in the universal enveloping algebra. For complex groups G, one of the main results in **[24]** tells us that, under certain assumptions on $\mathcal{O}_G$ and on π, the space of $C^{-\infty}$-generalized Whittaker vectors of type $\mathcal{O}_G$ is nonzero and finite-dimensional if and only if the closure of $\mathcal{O}_G$ coincides with the wave front set of π.

As to p-adic groups, Mœglin and Waldspurger have already established in 1987 a stronger result of this nature, by showing that the wave front cycle (asymptotic cycle) of an irreducible representation π of G completely controls the spaces of generalized Whittaker vectors of interest. Namely, it is proved in **[26]** that, if $\mathcal{O}_G$ is a nilpotent orbit which is maximal in the wave front set (asymptotic support) of π, the dimension of the space of generalized Whittaker vectors of type $\mathcal{O}_G$ is equal to the multiplicity of $\mathcal{O}_G$ in the wave front cycle. However, up to this time, the corresponding phenomenon is not yet fully understood in the case of real groups, except for the representations with the largest Gelfand-Kirillov dimension (see **[23]** and **[25]**).

In this article, we focus our attention on the irreducible admissible (unitary) highest weight representations of real simple Lie groups. These are representations with rather small Gelfand-Kirillov dimensions. We reveal a structure of the spaces of generalized Whittaker models in relation to the associated cycles of highest weight modules.

Now, let G be a connected simple Lie group with finite center, and let K be a maximal compact subgroup of G. Assume that $K\backslash G$ is Hermitian symmetric. The Lie algebras of G and K are denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. We write $K_{\mathbb{C}}$ (resp. $\mathfrak{g}, \mathfrak{k}$) for the complexifications of K (resp. $\mathfrak{g}_0, \mathfrak{k}_0$) respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a complexified Cartan decomposition of $\mathfrak{g}$, and let θ denote the corresponding Cartan involution of $\mathfrak{g}$. The G-invariant complex structure on $K\backslash G$ gives a triangular decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ of $\mathfrak{g}$. Conventionally, the complexification in $\mathfrak{g}$ of any real vector subspace $\mathfrak{s}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{s}$ by dropping the subscript 0. We write $U(\mathfrak{m})$ (resp. $S(\mathfrak{v})$) for the universal enveloping algebra of a Lie algebra $\mathfrak{m}$ (resp. the symmetric algebra of a vector space $\mathfrak{v}$).

The group G of Hermitian type has a distinguished family of irreducible admissible Hilbert representations with highest weights. The Harish-Chandra module of such a G-representation is obtained as the unique simple quotient $L(\tau)$ of generalized Verma module induced from an irreducible representation (τ, V_τ) of K. Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{q} := \mathfrak{k} + \mathfrak{p}_+$ of $\mathfrak{g}$ by making $\mathfrak{p}_+$ act on V_τ trivially. We call τ the extreme K-type of $L(\tau)$.

The purpose of this paper is to describe the generalized Whittaker models for irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$. To be more precise, let $\{\mathcal{O}_m \mid m = 0, \ldots, r\}$ be the totality of nilpotent $K_{\mathbb{C}}$-orbits in the nilradical $\mathfrak{p}_+$ of $\mathfrak{q}$, arranged as $\dim \mathcal{O}_0 = 0 < \dim \mathcal{O}_1 < \cdots < \dim \mathcal{O}_r = \dim \mathfrak{p}_+$. We write $\mathcal{O}'_m$ for the the nilpotent G-orbit in $\mathfrak{g}_0$ corresponding to $\mathcal{O}_m$ by the Kostant-Sekiguchi bijection. Following the recipe by Kawanaka [**14**] (see also [**40**]), we can construct a generalized Gelfand-Graev representation $\Gamma_m = \mathrm{Ind}_{\mathfrak{n}(m)}^{G}(\eta_m)$ (GGGR for short; see Definition 4.3) of G attached to $\mathcal{O}'_m$. On the other hand, it is well-known that the associated variety $\mathcal{V}(L(\tau))$ of a highest weight module $L(\tau)$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in $\mathfrak{p}_+$, where $m(\tau)$ depends on τ. Then our aim is to specify the $(\mathfrak{g}, K)$-embeddings of $L(\tau)$ into these GGGRs Γ_m $(m = 0, \ldots, r)$. This is a continuation of our earlier work [**41**] on Whittaker models for the holomorphic discrete series.

In order to specify the embeddings, we use the invariant differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to the K-representation τ^* dual to τ (Definition 2.3). This operator $\mathcal{D}_{\tau^*}$ is due to Enright, Davidson and Stanke ([**2**], [**3**], [**4**]). The K-finite kernel of $\mathcal{D}_{\tau^*}$ realizes the dual lowest weight module $L(\tau)^*$. By virtue of the kernel theorem given as Corollary 1.8, we find that the space $\mathcal{Y}(\tau, m)$ of η_m-covariant solutions of the differential equation $\mathcal{D}_{\tau^*}F = 0$ is isomorphic to the space of $(\mathfrak{g}, K)$-homomorphisms in question, where η_m is the character of nilpotent Lie subalgebra $\mathfrak{n}(m)$ of $\mathfrak{g}$ that defines Γ_m.

The space $\mathcal{Y}(\tau, m)$ can be intrinsically analyzed by means of the unbounded realization of $K\backslash G$ via the Cayley transform (cf. [**32**], [**9**]). Some remarkable results of

Enright and Joseph [**5**], Jakobsen [**20**] on the annihilator ideal of (unitarizable) highest weight modules are useful in the course of our study. Also, elementary properties (cf. Vogan [**33**, Section 2]) on the associated (characteristic) cycle of Harish-Chandra modules guarantee that the space $\mathcal{Y}(\tau, m)$ does not vanish for the most relevant $m = m(\tau)$. As a result, we get the following conclusions (see Theorems 4.7–4.9).

Theorem 1. — *$L(\tau)$ embeds into the GGGR Γ_m with nonzero and finite multiplicity if and only if the corresponding $\mathcal{O}_m$ is the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. In this case, the space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ consists only of elementary functions on the unbounded domain $\mathcal{S}\,(\subset \mathfrak{p}_-)$ which realizes $K\backslash G$.*

Theorem 2. — *If $L(\tau)$ is unitarizable, we can specify the space $\mathcal{Y}(\tau)$ in terms of the principal symbol at the origin Ke of the differential operator $\mathcal{D}_{\tau^*}$. This reveals a natural action on $\mathcal{Y}(\tau)$ of the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at a certain point $X \in \mathcal{O}_{m(\tau)}$. Furthermore, we find that the dimension of $\mathcal{Y}(\tau)$, that is, the multiplicity of embeddings $L(\tau) \hookrightarrow \Gamma_{m(\tau)}$, coincides with the multiplicity of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the defining ideal of $\mathcal{V}(L(\tau))$.*

For the classical groups $G = SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$, the theory of reductive dual pair gives explicit realizations of unitarizable highest weight modules $L(\tau)$ (cf. [**12**], [**7**], [**3**]). In this setting, it is not difficult to specify the generalized Whittaker models for such $L(\tau)$'s more explicitly by using the oscillator representation of a pair (G, G') with a compact group G' dual to G. In fact, this has been done by Tagawa [**31**] for the case $SU(p,q)$, motivated by author's observation in 1997 for the case $Sp(n,\mathbb{R})$. We include this observation as well as Tagawa's result at the end of this paper (see Theorems 5.14 and 5.15 together with the isomorphism (4.15)), handling all the groups $SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ in a unified manner.

The last statement in Theorem 2 clarifies the relationship between the generalized Whittaker models and the multiplicity in the associated cycle $\mathcal{AC}(L(\tau))$ of unitarizable highest weight module $L(\tau)$. In fact, $\mathcal{Y}(\tau)$ turns to be the dual of the isotropy representation of $K_{\mathbb{C}}(X)$ attached to $\mathcal{AC}(L(\tau))$ in the sense of Vogan [**33**]. We note that the associated cycle and the Bernstein degree of $L(\tau)$ have been specified by Nishiyama, Ochiai and Taniguchi [**27**] for the above classical groups G through detailed study of K-types of $L(\tau)$, where $L(\tau)$ is assumed to be an irreducible constituent of the oscillator representations of pairs (G, G') in the stable range (with smaller G'). Recently, Kato and Ochiai [**13**] have generalized the technique in [**27**] to a large extent. They established in particular a unified formula for the degrees of nilpotent orbits $\mathcal{O}_m$, which is valid for any simple Lie group of Hermitian type.

An η_m-equivariant linear form on $L(\tau)$ is called an (algebraic) generalized Whittaker vector of type η_m. Each $(\mathfrak{g}, K)$-embedding of $L(\tau)$ into the GGGR Γ_m, composed with the evaluation at the identity $e \in G$ of functions in Γ_m, naturally gives rise to a generalized Whittaker vector of type η_m on $L(\tau)$. We can show that the converse is also true for the most relevant case $m = m(\tau)$. Namely, it turns out that every generalized Whittaker vector of type η_m comes from a function in the space $\mathcal{Y}(\tau)$ for any $L(\tau)$ (see Proposition 4.19). This allows us to interpret the main results of this article in terms of algebraic generalized Whittaker vectors associated to irreducible highest weight $(\mathfrak{g}, K)$-modules (Theorem 4.22).

We organize this paper as follows.

Section 1 gives general theory on the embeddings of irreducible $(\mathfrak{g}, K)$-modules into induced G-representations. The kernel theorem (Corollary 1.8) is our main tool for studying generalized Whittaker models. We introduce in Section 2 the differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to τ^*, after [**4**]. In addition, the solutions F of $\mathcal{D}_{\tau^*}F = 0$ of exponential type are specified in Proposition 2.8. Section 3 is devoted to characterizing the associated variety and multiplicity of irreducible highest weight module $L(\tau)$ by means of the principal symbol of $\mathcal{D}_{\tau^*}$ (Theorem 3.11). In Section 4 we give our main results (Theorems 4.7–4.9) that describe the generalized Whittaker models for $L(\tau)$. Relation to algebraic generalized Whittaker vectors is also investigated. Last in Section 5, we discuss the case of classical groups $SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ more explicitly.

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1. Embeddings of Harish-Chandra modules

This section prepares some generalities about the embeddings of irreducible Harish-Chandra modules into C^∞-induced representations of a semisimple Lie group, by developing our earlier observation [**42**, I, §2] for the discrete series in full generality. The results stated in this section are more or less folklore for the experts, or they are consequences of some known facts concerning the maximal globalization of Harish-Chandra modules due to Schmid and Kashiwara (cf. [**29**], [**11**]). Nevertheless we include here the detail with direct proofs in order to keep this paper more accessible and self-contained. In fact, a kernel theorem, Corollary 1.8, will be essentially used in the succeeding sections to describe generalized Whittaker models for highest weight representations.

1.1. A duality of Peter-Weyl type. — Throughout this section, let G be any connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We keep the same notation and convention employed at the beginning of Introduction.