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Astérisque, tome 321 (2008), p. 99-128
<http://www.numdam.org/item?id=AST_2008__321__99_0>

Astérisque
321, 2008, p. 99–128

SPECIAL LAGRANGIAN FIBRATIONS, MIRROR SYMMETRY AND CALABI-YAU DOUBLE COVERS

by

Denis Auroux

To Jean Pierre Bourguignon on his 60th birthday, with my most sincere gratitude for the time he spent guiding me through the process of becoming a mathematician.

Abstract. — The first part of this paper is a review of the Strominger-Yau-Zaslow conjecture in various settings. In particular, we summarize how, given a pair (X, D) consisting of a Kähler manifold and an anticanonical divisor, families of special Lagrangian tori in $X \setminus D$ and weighted counts of holomorphic discs in X can be used to build a Landau-Ginzburg model mirror to X. In the second part we turn to more speculative considerations about Calabi-Yau manifolds with holomorphic involutions and their quotients. Namely, given a hypersurface H representing twice the anticanonical class in a Kähler manifold X, we attempt to relate special Lagrangian fibrations on $X \setminus H$ and on the (Calabi-Yau) double cover of X branched along H; unfortunately, the implications for mirror symmetry are far from clear.

***Résumé* (Fibrations lagrangiennes spéciales, symétrie miroir et revêtements doubles de Calabi-Yau)**

La première partie de cet article concerne la conjecture de Strominger-Yau-Zaslow dans diverses situations. En particulier nous décrivons comment, étant donnés une variété kählerienne X et un diviseur anticanonique D, un miroir de X dans la catégorie des modèles de Landau-Ginzburg peut être construit en considérant une famille de tores lagrangiens spéciaux dans $X \setminus D$ et en comptant des disques holomorphes dans X. La seconde partie est consacrée à des considérations plus spéculatives concernant les variétés de Calabi-Yau équipées d'une involution holomorphe et leurs quotients. Autrement dit, étant donnée une hypersurface H représentant le double de la classe anticanonique dans une variété kählerienne X, nous tentons d'établir un lien entre les fibrations lagrangiennes spéciales sur $X \setminus H$ et sur le revêtement double de X ramifié le long de H, qui est une variété de Calabi-Yau ; malheureusement, les conséquences pour la symétrie miroir sont loin d'être évidentes.

2000 Mathematics Subject Classification. — 14J32, 53D12.
Key words and phrases. — Mirror symmetry, special Lagrangian fibrations, Strominger-Yau-Zaslow conjecture, Calabi-Yau double covers.

Partially supported by NSF grants DMS-0600148 and DMS-0652630.

1. Introduction

The phenomenon of mirror symmetry was first evidenced for Calabi-Yau manifolds, i.e. Kähler manifolds with holomorphically trivial canonical bundle. Subsequently it became apparent that mirror symmetry also holds in a more general setting, if one enlarges the class of objects under consideration (see e.g. [**14**]); namely, one should allow the mirror to be a *Landau-Ginzburg model*, i.e. a pair consisting of a non-compact Kähler manifold and a holomorphic function on it (called *superpotential*).

Our motivation here is to understand how to *construct* the mirror manifold, starting from examples where the answer is known and extrapolating to less familiar situations; generally speaking, the verification of the mirror symmetry conjectures for the manifolds obtained by these constructions falls outside the scope of this paper.

The geometric understanding of mirror symmetry in the Calabi-Yau case relies on the Strominger-Yau-Zaslow (SYZ) conjecture [**28**], which roughly speaking postulates that mirror pairs of Calabi-Yau manifolds carry dual fibrations by special Lagrangian tori, and its subsequent refinements (see e.g. [**10, 21**]). This program can be extended to the non Calabi-Yau case, as suggested by Hori [**12**] and further investigated in [**3**]. In that case, the input consists of a pair (X, D) where X is a compact Kähler manifold and D is a complex hypersurface representing the anticanonical class. Observing that the complement of D carries a holomorphic n-form with poles along D, we can think of $X \setminus D$ as an open Calabi-Yau manifold, to which one can apply the SYZ program. Hence, one can attempt to construct the mirror of X as a (complexified) moduli space of special Lagrangian tori in $X \setminus D$, equipped with a Landau-Ginzburg superpotential defined by a weighted count of holomorphic discs in X. However, exceptional discs and wall-crossing phenomena require the incorporation of "instanton corrections" into the geometry of the mirror (see [**3**]).

One notable feature of the construction is that it provides a bridge between mirror symmetry for the Kähler manifold X and for the Calabi-Yau hypersurface $D \subset X$. Namely, the fiber of the Landau-Ginzburg superpotential is expected to be the SYZ mirror to D, and the two pictures of homological mirror symmetry (for X and for D) should be related via restriction functors (see Section 7 of [**3**] for a sketch).

In this paper, we would like to consider a slightly different situation, which should provide another relation with mirror symmetry for Calabi-Yau manifolds. The union of two copies of X glued together along D can be thought of as a singular Calabi-Yau manifold, which can be smoothed to a double cover of X branched along a hypersurface H representing twice the anticanonical class and contained in a neighborhood of D. This suggests that one might be able to think of mirror symmetry for X as a $\mathbb{Z}/2$-invariant version of mirror symmetry for the Calabi-Yau manifold Y. Unfortunately, this proposal comes with several caveats which make it difficult to implement.

Let (X, ω, J) be a compact Kähler manifold, and let H be a complex hypersurface in X representing twice the anticanonical class. Then the complement of H carries a nonvanishing section Θ of $K_X^{\otimes 2}$ with poles along H. We can think of Θ as the square of a holomorphic volume form defined up to sign. In this context, we can look for special Lagrangian submanifolds of $X \setminus H$, i.e. Lagrangian submanifolds on which the restriction of Θ is real. The philosophy of the SYZ conjecture suggests that, in favorable cases, one might be able to construct a foliation of $X \setminus H$ in which the generic leaves are special Lagrangian tori. Indeed, denote by Y the double cover of X branched along H: then Y is a Calabi-Yau manifold with a holomorphic involution. If Y carries a special Lagrangian fibration that is invariant under the involution, then by quotienting we could hope to obtain the desired foliation on $X \setminus H$; unfortunately the situation is complicated by technicalities involving the symplectic form.

Conjecture 1.1. — *For a suitable choice of H, $X \setminus H$ carries a special Lagrangian foliation whose lift to the Calabi-Yau double cover Y can be perturbed to a $\mathbb{Z}/2$-invariant special Lagrangian torus fibration.*

If $-K_X$ is effective, we can consider a situation where H degenerates to a hypersurface D representing the anticanonical class in X, with multiplicity 2. As explained above, this corresponds to the situation where Y degenerates to the union of two copies of X glued together along D. One could hope that under such a degeneration the foliation on $X \setminus H$ converges to a special Lagrangian torus fibration on $X \setminus D$. Using the mirror construction described in [**3**], one can then try to relate a Landau-Ginzburg mirror $(X^\vee, W)$ of X to a Calabi-Yau mirror $Y^\vee$ of Y. The simplest case should be when $K_{X|D}$ is holomorphically trivial (which in particular requires $c_1(X)^2 = 0$). Then $W : X^\vee \to \mathbb{C}$ is expected to have trivial monodromy around infinity (see Remark 2.11), so that $\partial X^\vee \approx S^1 \times D^\vee$ where $D^\vee$ is mirror to D. It is then tempting to conjecture that, considering only the complex structure of the mirror (and ignoring its symplectic geometry), $Y^\vee$ can be obtained by gluing together two copies of the mirror $X^\vee$ to X along their boundary $S^1 \times D^\vee$. Unfortunately, as we will see in § 3.5 this is not compatible with instanton corrections.

The rest of this paper is organized as follows. In Section 2 we review the geometry of mirror symmetry from the perspective of the SYZ conjecture, both in the Calabi-Yau case and in the more general case (relatively to an anticanonical divisor). We then turn to more speculative considerations in Section 3, where we discuss the geometry of Calabi-Yau double covers, clarify the statement of Conjecture 1.1, and consider various examples.

Acknowledgements. — I would like to thank Mohammed Abouzaid, Paul Seidel, Ludmil Katzarkov, and Dima Orlov for many fruitful discussions. I am also grateful to Ron Donagi, whose interest in this topic prompted the writing of this paper, and to the referee for insightful comments. This research was partially supported by NSF grants DMS-0600148 and DMS-0652630.

2. The SYZ conjecture and mirror symmetry

2.1. Motivation. — One of the most spectacular mathematical predictions of string theory is the phenomenon of mirror symmetry, i.e. the existence of a broad dictionary under which the symplectic geometry of a given manifold X can be understood in terms of the complex geometry of a mirror manifold $X^\vee$, and vice-versa. This dictionary works at several levels, among which perhaps the most exciting is Kontsevich's homological mirror conjecture, which states that the derived Fukaya category of X should be equivalent to the derived category of coherent sheaves of its mirror $X^\vee$ [**19**]; in the non Calabi-Yau case the categories under consideration need to be modified appropriately [**20**] (see also [**1, 13, 18, 26, 27**]).

The main goal of the Strominger-Yau-Zaslow conjecture [**28**] is to provide a geometric interpretation of mirror symmetry. Roughly speaking it says that mirror manifolds carry dual fibrations by special Lagrangian tori. In the Calabi-Yau case, one way to motivate the conjecture is to observe that, given any point p of the mirror $X^\vee$, mirror symmetry should put the skyscraper sheaf $\mathcal{O}_p$ in correspondence with some object $\mathcal{L}_p$ of the Fukaya category of X. As a graded vector space $\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p)$ is isomorphic to the cohomology of T^n; therefore the most likely candidate for $\mathcal{L}_p$ is a (special) Lagrangian torus in X, equipped with a rank 1 unitary local system (a flat $U(1)$ bundle). This suggests that one should try to construct $X^\vee$ as a moduli space of pairs (L, ∇) where L is a special Lagrangian torus in X and ∇ is a flat unitary connection on the trivial line bundle over L. Since for each torus L the moduli space of flat connections can be thought of as a dual torus, we arrive at the familiar picture.

When X is not Calabi-Yau but the anticanonical class $-K_X$ is effective, we can still equip the complement of a hypersurface $D \in |-K_X|$ with a holomorphic volume form, and thus consider special Lagrangian tori in $X \setminus D$. However, in this case, holomorphic discs in X with boundary in L cause Floer homology to be *obstructed* in the sense of Fukaya-Oh-Ohta-Ono [**6**]: to each object $\mathcal{L} = (L, \nabla)$ we can associate an obstruction $\mathfrak{m}_0(\mathcal{L})$, given by a weighted count of holomorphic discs in (X, L), and the Floer differential on $CF^*(\mathcal{L}, \mathcal{L}')$ squares to $\mathfrak{m}_0(\mathcal{L}') - \mathfrak{m}_0(\mathcal{L})$. Moreover, even when the Floer homology groups $HF^*(\mathcal{L}, \mathcal{L})$ can still be defined, they are often zero, so that $\mathcal{L}$ is a trivial object of the Fukaya category. On the mirror side, these features of the theory can be replicated by the introduction of a *Landau-Ginzburg superpotential*, i.e. a holomorphic function $W : X^\vee \to \mathbb{C}$. Without getting into details, W can be thought of as an obstruction term for the B-model on $X^\vee$, playing the same role as