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THE GENERALIZED HODGE AND BLOCH CONJECTURES ARE EQUIVALENT FOR GENERAL COMPLETE INTERSECTIONS

BY Claire VOISIN

ABSTRACT. – We prove that Bloch's conjecture is true for surfaces with $p_g = 0$ obtained as 0-sets X_σ of a section σ of a very ample vector bundle on a variety X with "trivial" Chow groups. We get a similar result in presence of a finite group action, showing that if a projector of the group acts as 0 on holomorphic 2-forms of X_σ, then it acts as 0 on 0-cycles of degree 0 of X_σ. In higher dimension, we also prove a similar but conditional result showing that the generalized Hodge conjecture for general X_σ implies the generalized Bloch conjecture for any smooth X_σ, assuming the Lefschetz standard conjecture (the last hypothesis is not needed in dimension 3).

RÉSUMÉ. – Nous montrons la conjecture de Bloch pour les surfaces avec $p_g = 0$ obtenues comme lieux des zéros X_σ d'une section σ d'un fibré vectoriel très ample sur une variété X à groupes de Chow « triviaux ». Nous obtenons un résultat similaire en présence d'une action d'un groupe fini, montrant que si un projecteur du groupe agit comme 0 sur les 2-formes holomorphes de X_σ, il agit comme 0 sur les 0-cycles de degré 0 de X_σ. En dimension supérieure, nous obtenons un résultat similaire mais conditionnel montrant que la conjecture de Hodge généralisée pour X_σ générale entraîne la conjecture de Bloch généralisée pour tout X_σ lisse, en supposant satisfaite la conjecture de Lefschetz standard (cette dernière hypothèse n'étant pas nécessaire en dimension 3).

1. Introduction

Recall first that a weight k Hodge structure $(L, L^{p,q})$ has Hodge coniveau $c \leq \frac{k}{2}$ if the Hodge decomposition of $L_{\mathbb{C}}$ takes the form

$$L_{\mathbb{C}} = L^{k-c,c} \oplus L^{k-c-1,c+1} \oplus \cdots \oplus L^{c,k-c}$$

with $L^{k-c,c} \neq 0$. If X is a smooth complex projective variety and $Y \subset X$ is a closed algebraic subset of codimension c, then $\mathrm{Ker}\,(H^k(X,\mathbb{Q}) \to H^k(X \setminus Y, \mathbb{Q}))$ is a sub-Hodge structure of coniveau $\geq c$ of $H^k(X,\mathbb{Q})$ (cf. [34, Theorem 7]). The generalized Hodge conjecture formulated by Grothendieck [10] is the following.

Conjecture 1.1. – *Let X be as above and let $L \subset H^k(X,\mathbb{Q})$ be a sub-Hodge structure of Hodge coniveau $\geq c$. Then L has* geometric *coniveau $\geq c$, i.e., there exists a closed algebraic subset $Y \subset X$ of codimension c such that*

$$L \subset \mathrm{Ker}\,(H^k(X,\mathbb{Q}) \to H^k(X \setminus Y,\mathbb{Q})).$$

This conjecture is widely open, even for general hypersurfaces or complete intersections in projective space (cf. [33]). Consider a smooth complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degrees $d_1 \leq \cdots \leq d_r$. Then the Hodge coniveau of the Hodge structure on $H^{n-r}(X,\mathbb{Q})_{\mathrm{prim}}$ (the only part of the cohomology of X which does not come from the ambient space) is given by the formula (cf. [33], where complete intersections of coniveau 2 are studied):

$$\text{coniveau}(H^{n-r}(X,\mathbb{Q})_{\mathrm{prim}}) \geq c \Leftrightarrow n \geq \sum_i d_i + (c-1)d_r. \tag{1}$$

A deep relation between Chow groups and geometric coniveau appears in the various generalizations of Mumford's theorem obtained in [3], [26], [16], [14], [22], based on refinements of the diagonal decomposition principle due to Bloch and Srinivas. The resulting statement is the following (see [32, II,10.3.2]):

Theorem 1.2. – *Let X be a smooth projective variety of dimension m. Assume that the cycle class map*

$$\mathrm{cl} : CH_i(X)_{\mathbb{Q}} \to H^{2m-2i}(X,\mathbb{Q})$$

is injective for $i \leq c-1$. Then for any $k \geq 0$, the geometric coniveau of the Hodge structure $H^k(X,\mathbb{Q})^{\perp \mathrm{alg}}$ is greater than or equal to c.

Here $H^k(X,\mathbb{Q})^{\perp \mathrm{alg}}$ denotes the "transcendental part" of the cohomology, that is, the set of classes which are orthogonal to all cycle classes via Poincaré duality. It is of course different from $H^k(X,\mathbb{Q})$ only if k is even.

This theorem could be a possible way to attack Conjecture 1.1 for the whole cohomology of X, or at least its transcendental part. The first case of this theorem, that is the case where $c = 1$, was obtained by Bloch-Srinivas [3]. It says that if a variety X has $CH_0(X) = \mathbb{Z}$, then $H^{k,0}(X) = 0$ for any $k > 0$ (which generalizes Mumford's theorem [17]) and furthermore, the cohomology of positive degree of X is supported on a proper algebraic subset $Y \subset X$ (which solves Conjecture 1.1 for such X and for coniveau 1).

The next major open problem, which by the above theorem would solve the generalized Hodge conjecture, is the following conjecture relating the Hodge coniveau and Chow groups. This conjecture is a vast generalization of Bloch conjecture for surfaces [2].

Conjecture 1.3 (cf. [32, II, 11.2.2]). – *Let X be a smooth projective variety of dimension m satisfying the condition $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$). Then for any integer $i \leq c-1$, the cycle class map*

$$\mathrm{cl} : CH_i(X)_{\mathbb{Q}} \to H^{2m-2i}(X,\mathbb{Q})$$

is injective.

Note that assuming the Hodge conjecture, the assumptions are equivalent to saying that the cohomology $H^*(X,\mathbb{Q})^{\perp\mathrm{alg}}$ has Hodge coniveau $\geq c$.

If we look at the case of hypersurfaces or complete intersections in projective space, we see from (1) that Conjecture 1.3 predicts the following :

Conjecture 1.4. – *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degrees $d_1 \leq \cdots \leq d_r$. Then if $n \geq \sum_i d_i + (c-1)d_r$, the cycle class map*

$$\mathrm{cl} : CH_i(X)_{\mathbb{Q}} \to H^{2n-2r-2i}(X,\mathbb{Q})$$

is injective for any integer $i \leq c-1$.

Note that, by Theorem 1.2, this conjecture would imply Conjecture 1.1 for very general complete intersections, that is, Conjecture 1.5 below. Indeed, by monodromy arguments, the Hodge structure on the primitive middle cohomology of a very general complete intersection is simple except for some rare and classified cases where it is made of Hodge classes. Thus in this case a nontrivial sub-Hodge structure must be the whole primitive cohomology, and its Hodge coniveau is computed by (1).

Apart from very particular values of the degrees d_i (like complete intersections of quadrics [21], or cubics of small dimension [1]), Conjecture 1.4 is essentially known only in the case $c = 1$, where the considered complete intersections are Fano, hence rationally connected, so that the equality $CH_0(X) = \mathbb{Z}$ is trivial in this case.

In the paper [31], it is proved that for any pair (n,d), there are smooth hypersurfaces of degree d in $\mathbb{P}^n$ satisfying the conclusion of Conjecture 1.4.

Coming back to Conjecture 1.1 for general complete intersections in projective space, we get from (1) that it is equivalent to the following statement:

Conjecture 1.5. – *The primitive cohomology $H^{n-r}(X,\mathbb{Q})_{\mathrm{prim}}$ of a smooth complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degrees $d_1 \leq \cdots \leq d_r$ vanishes on the complement of a closed algebraic subset $Y \subset X$ of codimension c if $n \geq \sum_i d_i + (c-1)d_r$.*

As already mentioned, Conjecture 1.5 would be implied by Conjecture 1.4 using Theorem 1.2. The paper [33] is an attempt to prove directly Conjecture 1.5 for hypersurfaces or complete intersections of coniveau 2 without trying to show the triviality of their CH_0 and CH_1 groups. Conjecture 1.5 is shown there to be implied by a conjecture on the effective cone of algebraic cycles (on some auxiliary variety). This work was motivated by the fact that, apart from the case of coniveau 1, as is apparent from the lack of progress in this direction and the fact that the results obtained this way (see [7], [22], [20]) cover a numerical range which is rather different from the one predicted by Conjecture 1.5, it seems now unlikely that one will prove Conjecture 1.5 by an application of Theorem 1.2, that is via the proof of the triviality of Chow groups of small dimension.

In fact, we will essentially show in this paper that for very general complete intersections of ample hypersurfaces or more generally, zero sets of sections of very ample vector bundles inside any smooth projective variety X with "trivial" Chow groups, Conjecture 1.1 (that is Conjecture 1.5 if the ambient space is $\mathbb{P}^n$) implies Conjecture 1.3 (that is Conjecture 1.4 if the ambient space is $\mathbb{P}^n$). Stated this way, this is not completely correct, and we have to add an extra assumption that we now explain.

Let us state the following conjecture, that we will relate later on (cf. Proposition 2.6) to the so-called standard conjectures [13]:

CONJECTURE 1.6. – *Let X be a smooth complex projective variety, and let $Y \subset X$ be a closed algebraic subset. Let $Z \subset X$ be a codimension k algebraic cycle, and assume that the cohomology class $[Z] \in H^{2k}(X, \mathbb{Q})$ vanishes in $H^{2k}(X \setminus Y, \mathbb{Q})$. Then there exists a codimension k cycle Z' on X with $\mathbb{Q}$-coefficients, which is supported on Y and such that $[Z'] = [Z]$ in $H^{2k}(X, \mathbb{Q})$.*

Our main result in this paper is the following theorem. We will say here that a smooth variety X has trivial Chow groups if the cycle map $\mathrm{cl} : CH^i(X)_{\mathbb{Q}} \to H^{2i}(X, \mathbb{Q})$ is injective for any i. In the theorem below, we say that a vector bundle E on X is very ample if for any subscheme $z \subset X$ of length 2, the restriction map $H^0(X, E) \to H^0(z, E_{|z})$ is surjective.

THEOREM 1.7. – *Assume Conjecture 1.6 holds for cycles of codimension $n - r$. Let X be a smooth complex projective n-fold with trivial Chow groups. Let E be a very ample vector bundle on X of rank r. Assume that for a very general subvariety $X_b \subset X$ defined as the zero locus of a section of E, the Hodge structure on $H^{n-r}(X_b, \mathbb{Q})_{\mathrm{van}}$ is supported on a closed algebraic subset $Y_b \subset X_b$ of codimension $\geq c$. Then for the general such X_b (hence in fact for all), the cycle map $\mathrm{cl} : CH_i(X_b)_{\mathbb{Q}} \to H^{2n-2r-2i}(X_b, \mathbb{Q})$ is injective for any $i < c$.*

Here the space $H^{n-r}(X_b, \mathbb{Q})_{\mathrm{van}}$ of vanishing cohomology is defined as

$$\mathrm{Ker}\,(j_{b*} : H^{n-r}(X_b, \mathbb{Q}) \to H^{n+r}(X, \mathbb{Q})),$$

where j_b is the inclusion of X_b in X. If the ambient space X is $\mathbb{P}^n$, the vanishing cohomology is nothing but the primitive cohomology with respect to the line bundle $\mathcal{O}_{X_b}(1)$.

As a particular case, we get:

COROLLARY 1.8. – *Assuming Conjecture 1.6 for cycles of codimension $n - r$, the generalized Hodge Conjecture 1.5 implies the generalized Bloch Conjecture 1.4 for complete intersections in projective space.*

These results are conditional results. However, in small dimension, some assumptions are automatically satisfied, and this gives us the following statement, which will be proved in Subsection 3.3.

THEOREM 1.9. – *Let X be a smooth complex projective variety of dimension n with trivial Chow groups. Let E be a very ample vector bundle of rank r on X. Then we have:*

1) Case $n - r = 2$. *If the smooth surfaces X_b obtained as zero sets of sections of E have $h^{2,0}(X_b) = 0$, then they satisfy $CH_0(X_b) = \mathbb{Z}$. (This is the Bloch conjecture).*

2) Case $n - r = 3$. *If very general threefolds X_b obtained as zero sets of sections of E have the property that the degree 3 cohomology of X_b is of geometric coniveau 1 (which is also equivalent to the fact that Abel-Jacobi map $CH^2(X_b)_{\mathrm{hom}} \to J^3(X_b)$ is surjective), then the Chow group $CH_0(X_b)$ is equal to $\mathbb{Z}$.*