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NORM FORMS FOR ARBITRARY NUMBER FIELDS AS PRODUCTS OF LINEAR POLYNOMIALS

BY TIM D. BROWNING AND LILIAN MATTHIESEN

ABSTRACT. – Given a number field $K/\mathbb{Q}$ and a polynomial $P \in \mathbb{Q}[t]$, all of whose roots are in $\mathbb{Q}$, let X be the variety defined by the equation $\mathbf{N}_K(\mathbf{x}) = P(t)$. Combining additive combinatorics with descent we show that the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on any smooth and projective model of X.

RÉSUMÉ. – Étant donnés un corps de nombres $K/\mathbb{Q}$ et un polynôme $P \in \mathbb{Q}[t]$, dont toutes les racines sont dans $\mathbb{Q}$, soit X la variété définie par l'équation $\mathbf{N}_K(\mathbf{x}) = P(t)$. En combinant la combinatoire additive avec la descente, nous montrons que l'obstruction Brauer–Manin est le seul obstacle au principe de Hasse et à l'approximation faible sur un modèle projectif et lisse de X.

1. Introduction

Let $K/\mathbb{Q}$ be a finite extension of number fields of degree $n \geqslant 2$ and fix a basis $\{\omega_1, \dots, \omega_n\}$ for K as a vector space over $\mathbb{Q}$. We will denote by

$$\mathbf{N}_K(x_1, \dots, x_n) = N_{K/\mathbb{Q}}(x_1\omega_1 + \cdots + x_n\omega_n)$$

the corresponding norm form, where $N_{K/\mathbb{Q}}$ denotes the field norm. The objective of this paper is to study the Hasse principle and weak approximation for the class of varieties $X \subset \mathbb{A}^{n+1}$ satisfying the Diophantine equation

$$P(t) = \mathbf{N}_K(x_1, \dots, x_n), \tag{1.1}$$

where $P(t)$ is a product of linear polynomials all defined over $\mathbb{Q}$. If r denotes the number of distinct roots of P, then P takes the form

$$P(t) = c^{-1} \prod_{i=1}^{r} (t - e_i)^{m_i}, \tag{1.2}$$

for $c \in \mathbb{Q}^*$, $m_1, \dots, m_r \in \mathbb{Z}_{>0}$ and pairwise distinct $e_1, \dots, e_r \in \mathbb{Q}$.

We let X^c be a smooth and projective model of X. Such a model X^c need not satisfy the Hasse principle and weak approximation, as has been observed by Coray (see [5, Eq. (8.2)]).

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Specifically, when $P(t) = t(t-1)$ and K is the cubic extension $\mathbb{Q}(\theta)$, obtained by adjoining a root θ of $x^3 - 7x^2 + 14x - 7 = 0$, then the set $X^c(\mathbb{Q})$ is not dense in $X^c(\mathbb{Q}_7)$. It has, however, been conjectured by Colliot-Thélène (see [3]) that all counter-examples to the Hasse principle and weak approximation for X^c are accounted for by the Brauer–Manin obstruction.

This conjecture covers the more general case where X arises from an equation of the form (1.1), but the ground field may be an arbitrary number field k instead of $\mathbb{Q}$ and the polynomial need not factorize completely over k. In this more general setting the problem of establishing Colliot-Thélène's conjecture has been addressed under various assumptions on the extension K/k and upon the polynomial $P(t)$. Thus the conjecture is now known to be true for Châtelet surfaces ($[K : k] = 2$ and $\deg(P(t)) \leqslant 4$) by work of Colliot-Thélène, Sansuc and Swinnerton-Dyer [8, 9], a family of singular cubic hypersurfaces ($[K : k] = 3$ and $\deg(P(t)) \leqslant 3$) by work of Colliot-Thélène and Salberger [5], the case where K/k is arbitrary and $P(t)$ is split over k with at most two distinct roots (see [4, 22, 30, 33]) and the case where $K/\mathbb{Q}$ is arbitrary and $P(t)$ is an irreducible quadratic polynomial over $\mathbb{Q}$ (see [1, 12]). Finally, if one assumes Schinzel's hypothesis, then it is true for K/k cyclic and $P(t)$ arbitrary, by work of Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [10].

Suppose now that $k = \mathbb{Q}$ and P is given by (1.2). Until recently, Colliot-Thélène's conjecture was only known to hold unconditionally when $r \leqslant 2$. When $r \leqslant 1$, the variety X is a principal homogeneous space for the algebraic torus $R^1_{K/\mathbb{Q}}$, and so the conjecture follows from work of Colliot-Thélène and Sansuc [6]. When $r = 2$, Heath-Brown and Skorobogatov [22] prove it under the additional assumption that $\gcd(n, m_1, m_2) = 1$, while Colliot-Thélène, Harari and Skorobogatov [4, Thm. 3.1] establish it in general. Our primary result establishes the conjecture for any $r \geqslant 1$.

THEOREM 1.1. – *The Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on* X^c.

By combining Theorem 1.1 with the Brauer group calculation in [4, Cor. 2.7] we obtain the following corollary.

COROLLARY 1.2. – *Suppose that* $\gcd(m_1, \ldots, m_r) = 1$ *and* K *does not contain a proper cyclic extension of* $\mathbb{Q}$. *Then* $X^c(\mathbb{Q}) \neq \emptyset$ *and* X^c *satisfies weak approximation.*

In [22], for the first time, Heath-Brown and Skorobogatov combined the descent theory of Colliot-Thélène and Sansuc [7] with the Hardy–Littlewood circle method, in order to study the Hasse principle and weak approximation. In joint work with Skorobogatov [2], we introduced additive combinatorics into this subject and showed how it may usefully be combined with descent. This approach allowed us to study the variety X when $K/\mathbb{Q}$ is quadratic. The case $n = 2$ of Theorem 1.1 is a special case of [2, Thm 1.1]. Subsequently, Harpaz, Skorobogatov and Wittenberg [21] succeeded in showing how the finite complexity case of the generalized Hardy–Littlewood conjecture for primes, as established by Green and Tao [17] and Green–Tao–Ziegler [20], can be used in place of Schinzel's hypothesis to study rational points on varieties using fibration arguments. Their work [21, Cor. 4.1] leads to a version of Theorem 1.1 in which the extension $K/\mathbb{Q}$ is assumed to be cyclic, a fact that was previously only available under Schinzel's hypothesis, as a special case of work by Colliot-Thélène and Swinnerton-Dyer [11] on pencils of Severi–Brauer varieties. Building on work of Wei [34], they also handle (see [21, Thm. 4.6]) the case in which K is a non-cyclic extension of $\mathbb{Q}$ of prime degree such that the Galois group of the normal closure of K over $\mathbb{Q}$ has a non-trivial abelian quotient. We emphasize that the results of the present paper are unconditional and make no assumptions on the degree of the field extension, nor upon the type of the extension, other than that the ground field is $\mathbb{Q}$.

Our approach is based upon the strategy of [2]. We use descent theory to reduce Theorem 1.1 to establishing the Hasse principle and weak approximation for some auxiliary varieties, which can be analyzed using additive combinatorics. To introduce these varieties, let

$$f_1, \dots, f_r \in \mathbb{Q}[u_1, \dots, u_s]$$

be a system of pairwise non-proportional homogeneous linear polynomials, with $s \geqslant 2$. For each $1 \leqslant i \leqslant r$, let K_i denote a number field of degree $n_i = [K_i : \mathbb{Q}] \geqslant 2$. Central to our investigation will be the smooth variety $\mathcal{V} \subset \mathbb{A}_{\mathbb{Q}}^{n_1+\cdots+n_r+s}$, defined by

$$0 \neq \mathbf{N}_{K_i}(\mathbf{x}_i) = f_i(u_1, \dots, u_s), \quad (1 \leqslant i \leqslant r), \tag{1.3}$$

where $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i})$. For this variety we establish the following theorem, whose proof forms the bulk of this paper.

THEOREM 1.3. – *The variety $\mathcal{V}$ defined by* (1.3) *satisfies the Hasse principle and weak approximation.*

In fact (see Theorem 5.2) we shall produce an asymptotic formula for the number of suitably constrained integral points on $\mathcal{V}$ of bounded height. When $\mathcal{V}$ only involves quadratic extensions, Theorem 1.3 recovers [2, Thm. 1.2]. The latter result was established using work of the second author [26, 27]. We will build on this work in order to obtain the general case of Theorem 1.3. When $K_1, \dots, K_r$ are all assumed to be cyclic extensions of $\mathbb{Q}$, a shorter proof of Theorem 1.3 can be found in [21, Thm. 1.3].

1.1. Overview

We indicate how Theorem 1.3 implies Theorem 1.1 at the end of this introduction. The remainder of this paper is organized as follows. The overall goal is to prove Theorem 1.3 by asymptotically counting points of bounded height in $\mathcal{V}(\mathbb{Z})$, taking into account the additional constraints that are imposed by the weak approximation conditions. The associated counting function is introduced in Section 5. The asymptotic formula obtained in Theorem 5.2 for this counting function may prove to be of independent interest. Theorem 5.2 is proved using Green and Tao's nilpotent Hardy–Littlewood method (see [17]) in combination with the Green–Tao–Ziegler inverse theorem [20].

While containing mostly classical material, Section 2 fixes the notation for the rest of the paper and describes a certain fundamental domain that is specific to our counting problems. Section 3 contains a variety of technical results required at later stages in the paper and may be consulted as needed. Section 4 studies the number of solutions to a congruence $\mathbf{N}_K(\mathbf{x}) \equiv A \pmod{p^m}$. These results are used in Section 5 in order to analyze the non-archimedean local densities that appear in the statement of Theorem 5.2. Section 6 establishes those estimates for the Green–Tao method that correspond to the minor arc estimates in the classical Hardy–Littlewood method. These are the estimates needed in order to