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CHAOTIC BEHAVIOUR IN THE SOLAR SYSTEM [following J. LASKAR]

by **Stefano MARMI**

0. INTRODUCTION

I. Newton certainly believed that the Solar System is topologically unstable. In his view the perturbations among the planets were strong enough to destroy the stability of the Solar System. He even made the hypothesis that God controls the instabilities so as to insure the existence of the Solar System: "but it is not to be conceived that mere mechanical causes could give birth to so many regular motions .... This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent powerful Being" [N, p. 544]. The problem of Solar System stability was (and for many aspects still is) a real one: Halley was able to show, by analyzing the Chaldean observations transmitted by Ptolemy, that Saturn was moving away from the Sun while Jupiter was getting closer. A crude extrapolation leads to a possible collision in 6 million years (Myrs) in the past.

From a mathematical point of view arguments supporting the long–time stability of the orbits of the planets were given by Lagrange, Laplace and Poisson who proved the absence of secular evolution (polynomial increase in time) of the semi–major axis of the planets up to third order in the planetary masses.

On the contrary the researches of Poincaré [P] and Birkhoff [B] showed that instabilities might occur in the dynamics of the planets and that the phase space must have a quite complicated structure.

In the course of the year 1954, Kolmogorov [K] stated his famous theorem of persistence of quasiperiodic motions in near to integrable Hamiltonian systems, and first suggested that the picture may be twofold: stability in the sense of measure theory conjugated with topological instability. Arnold moreover proved [Ar1] that bounded orbits have positive measure in the planar three–body problem and claimed that the same result must be true for the n–body problem (provided that the masses of the planets are sufficiently – unrealistically – small). He also first proved in the general context of Hamiltonian systems with many degrees of freedom the existence of orbits which drift (or diffuse) along resonances so as to change by a finite amount their action [Ar2][1], even if this process is very slow [Ne]. This reinforced the belief that "the time after which chaos manifests itself under a sufficiently small perturbation of the initial state is large in comparison with the time of existence of the Solar System" [Ar3, p.82].

Following Herman [He2] one can legitimately ask the following question: *If one of the masses $m_0 = 1$ and all the other masses $m_j \ll 1$ are sufficiently small, are there wandering domains*[2] *in any neighborhood of fixed distinct circular orbits around the mass m_0 and moving in the same direction in a plane?*

Quite recently some progress has been made in the heuristic understanding of the dynamics of the planets of the Solar System, due largely to the help provided by computers but also due to a better understanding of the underlying dynamics, resulting from the great progress in the overall field of Dynamical Systems. Modern computers allow extensive analytic calculations and numerical integrations of realistic models over very long times, even if the shortness of the step–size needed for the computation has for many years limited the investigation to the outer planets of the Solar System (Jupiter, Saturn, Uranus, Neptune and Pluto) [CMN], [SW]. Indeed, the faster the orbital movement of the planet is, the shorter is the step–size required (from approximately 40 days for Jupiter to 12 hours for Mercury). As a result, until 1991 the only available numerical integration of a realistic model of the full Solar System was spanning only 44 centuries. For this reason the analytical approach, which makes use of perturbation theory, is needed.

J. Laskar replaced the full Newtonian equations of the motion by the so–called *secular system* introduced by Lagrange where the fast angular variables are eliminated. This system, instead of giving the fast motion of the planets in space, describes the slow deformation of the planets' orbits. In this way Laskar reduced the number of degrees of freedom of the system and achieved an impressing reduction of the step–size required (most of the computational time in traditional numerical integrations is actually spent in the numerical solution of Kepler's problem). In fact he was able to

[1] "Thus, even if the motion of a planet or an asteroid is regular, an arbitrarily small perturbation of the initial state is sufficient to make it chaotic" [Ar3, p.82].

[2] i.e. an open set V and $t_0 > 0$ such that $f^t(V) \cap V = \emptyset$ for all $t > t_0$, where f^t denotes the Hamiltonian flow.

use a step–size of 500 years. The numerical integration of this system shows that *the inner Solar System* (Mercury, Venus, Earth and Mars) *is chaotic* with a Lyapounov time of 5 Myrs. This measures the rate of the exponential growth of the distance in phase space between the orbits of two points initially close [Yo]. As a consequence it is not possible to compute ephemeris for the position of the Earth over 100 Myrs: an error of 15 meters in the initial position of the Earth may grow to an error of 150 million kilometers (i.e. its present distance to the Sun) after 100 Myrs. This kind of strong instability could even result in the escape of Mercury in 3.5 billion years (Gyrs). The deformations of the planets' orbits is responsible for an external forcing on the Hamiltonian describing the evolution of the obliquity of each planet. The obliquity is the angle between the equator and the orbital plane. Laskar shows that it can undergo dramatic variations on a time scale very short in geological terms.

In what follows we will describe these results and the ideas underlying Laskar's approach, mainly coming from the theory of Hamiltonian systems and classical perturbation theory. We will also briefly discuss the technique of the numerical analysis of the fundamental frequency developed by Laskar to study the mixed phase space structure of quasi–integrable Hamiltonian systems.

The style of this exposition will be quite informal, partially because most of the results reported here lack a rigorous justification (and sometimes even a good mathematical formulation). In the last section we will try to formulate some open problems inspired by Laskar's work.

Acknowledgments. In the preparation of this review I have extensively used references [Ar1], [L1995a], [L1996] and some unpublished seminar notes of M. Herman [He3]. I have also benefited a lot from many discussions with A. Chenciner, J. Laskar, D. Sauzin and J.–C. Yoccoz.

0.1. Hamiltonian systems, integrable systems, quasiperiodic motions [Ar4]

Usually in mechanics the equations of the motion of a conservative system with phase space $M = T^*N$ (here the configuration space N is an f–dimensional riemannian manifold) are given in Hamiltonian form: $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $1 \leq i \leq f$. Here the "generalized coordinates" q_i and their "conjugate momenta" p_i are a system of local canonical coordinates in M and $H : M \to \mathbb{R}$ is smooth (the Hamiltonian of the system). Note that in many problems arising from celestial mechanics the flow is not complete due to the unavoidable occurence of collisions (see [Ch2] for a recent review). The symplectic form on M is $\omega = \sum_{i=1}^{f} dp_i \wedge dq_i$ and maintains this expression in all canonical systems of coordinates (they form an atlas by Darboux' theorem). Two functions $F, G \in \mathcal{C}^\infty(M, \mathbb{R})$ are *in involution* if their Poisson bracket $\{F, G\} = 0$, i. e. when their Hamiltonian flows commute.

An important extension of the Hamiltonian formalism is obtained considering time–dependent Hamiltonian functions $H : M \times \mathbb{R} \to \mathbb{R}$. These are especially useful, as we will see, for modeling non–isolated systems, i. e. mechanical systems under the action of some external forcing.

An especially interesting case is provided by the manifold $\mathbb{R}^f \times \mathbb{T}^f$ which can be identified with the cotangent bundle of the f–dimensional torus $\mathbb{T}^f = \mathbb{R}^f/(2\pi\mathbb{Z})^f$. This manifold has a natural symplectic structure defined by the closed 2–form $\omega = \sum_{i=1}^f dJ_i \wedge d\vartheta_i$ where $(J_1, \dots J_f, \vartheta_1, \dots \vartheta_f)$ is a point on $\mathbb{R}^f \times \mathbb{T}^f$. Let U denote an open connected subset of $\mathbb{R}^f$. Whenever an Hamiltonian system can be reduced by a symplectic change of coordinates to a function $H : U \times \mathbb{T}^f \to \mathbb{R}$ which *does not depend on the angular variables* ϑ one says that the system is *completely canonically integrable* and the variables J are called *action variables.* Note that in this case Hamilton's equations take the particularly simple form $\dot{J}_i = -\frac{\partial H}{\partial \vartheta_i} = 0$, $\dot{\vartheta}_i = \frac{\partial H}{\partial J_i}$, $i = 1, \dots, f$. Let $\nu_i(J) = \frac{\partial H}{\partial J_i}$, $i = 1, \dots, f$. The associated flow $t \mapsto (J(t) \equiv J(0)$, $\vartheta(t) = \vartheta(0) + t\nu(J(0)))$, $t \in \mathbb{R}$, is *linear* and leaves invariant the f–dimensional torus $J = J(0)$. The motion is therefore bounded and *quasiperiodic* if the $\mathbb{Z}$–module $\{k \in \mathbb{Z}^f, k \cdot \nu(J(0)) = 0\}$ has dimension at most $f - 2$. Otherwise the motion is periodic. The Hamiltonian is *non degenerate* (i.e. satisfies the "twist condition") if $\det\left(\frac{\partial^2 H}{\partial J_i \partial J_k}(J)\right) \neq 0$ for all $J \in U$, thus the "frequency map"

$$J \mapsto \nu(J) = \frac{\partial H}{\partial J}(J) \in \mathbb{R}^f \tag{0.1}$$

is a local diffeomorphism. This condition is generic, but in many applications and especially in the two–body problem (see below) the Hamiltonian H is *properly degenerate*: $\det\left(\frac{\partial^2 H}{\partial J_i \partial J_k}(J)\right) = 0$ for all $J \in U$. In this case the linear flow on the invariant tori can be described by only $f_0 < f$ frequencies, where $f_0 = \operatorname{rank}\left(\frac{\partial^2 H}{\partial J_i \partial J_k}\right)$, by suitably choosing new action–angle coordinates $J' = (A^T)^{-1}J$, $\vartheta' = A\vartheta$, $A \in \mathrm{GL}(f, \mathbb{Z})$.

0.2. The two–body problem and action–angle variables. If the mutual attraction of the planets is neglected, each planet is attracted only by the Sun. This leads to the two–body problem whose Hamiltonian $\mathcal{H}_0 : T^*(\mathbb{R}^3 \setminus \{0\}) \mapsto \mathbb{R}$ in the center of