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CALABI–YAU THREEFOLDS OF BORCEA–VOISIN, ANALYTIC TORSION, AND BORCHERDS PRODUCTS

by

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Dedicated to Professor Jean-Michel Bismut on his sixtieth birthday

Abstract. — For a class of Borcea–Voisin threefolds, we give an explicit formula for the BCOV invariant [**3**], [**14**] as a function on the moduli space. For those Calabi–Yau threefolds, the BCOV invariant is expressed as the Petersson norm of the tensor product of a certain Borcherds lift on the Kähler moduli of a Del Pezzo surface and the Dedekind η-function. As a by-product, we construct an automorphic form on the orthogonal modular variety associated to the odd unimodular lattice of signature $(2, m)$, $m \leq 10$, which vanishes exactly on the Heegner divisor of norm (-1)-vectors.

Résumé **(Variétés de Calabi-Yau de dimension trois de type Borcea–Voisin, torsion analytique, et produits de Borcherds)**

Pour une classe de variétés de Borcea–Voisin, nous donnons une formule explicite de l'invariant de BCOV [**3**], [**14**] comme une fonction sur l'espace de modules. Pour ces variétés de Calabi–Yau de dimension trois, l'invariant de BCOV s'exprime comme la norme du produit tensoriel d'un relèvement de Borcherds à l'espace des modules kählériens d'une surface de Del Pezzo et de la fonction η de Dedekind. Nous construisons une forme automorphe sur la variété modulaire orthogonale associée au réseau unimodulaire impair de signature $(2, m)$, $m \leq 10$, qui s'annule exactement sur le diviseur de Heegner des vecteurs de norme -1.

1. Introduction

In [**33**], Ray–Singer introduced the notion of analytic torsion for compact Kähler manifolds. Their definition was extended to arbitrary holomorphic Hermitian vector bundles over a compact Kähler manifold by Quillen [**32**] and Bismut–Gillet–Soulé [**7**]. Let $\xi \to X$ be a holomorphic Hermitian vector bundle over a compact Kähler

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manifold and let $\zeta_q(s)$ be the spectral zeta function of the Hodge–Kodaira Laplacian acting on the space of $(0,q)$-forms on X with values in ξ. Then the real number

$$\tau(X,\xi) = \exp[-\sum_{q\geq 0}(-1)^q q\,\zeta_q'(0)]$$

is called the analytic torsion of ξ. The most fundamental results in the theory of analytic torsion such as the first variational formula, the second variational formula and the comparison formula for complex immersions were obtained by Bismut–Gillet–Soulé and Bismut–Lebeau as the corresponding results in the theory of Quillen metrics, i.e., the anomaly formula, the curvature formula and the immersion formula for Quillen metrics [**7**], [**8**],...

In [**3**], Bershadsky–Cecotti–Ooguri–Vafa introduced the following combination of analytic torsions for a compact Kähler manifold X

$$\prod_{p\geq 0}\tau(X,\Omega_X^p)^{(-1)^p p},$$

which we call the BCOV torsion of X. They studied the BCOV torsion as a function on the moduli space of Calabi–Yau threefolds and used it to extend the mirror symmetry conjecture to higher-genus Gromov–Witten invariants [**2**], [**3**].

In [**14**], the notion of BCOV invariant was introduced for Calabi–Yau threefolds by Fang–Lu–Yoshikawa, which they obtained using the BCOV torsion and a certain Bott–Chern secondary class. (See Sect. 5.1 for the definition.) The BCOV invariant of a Calabi–Yau threefold X is denoted by $\tau_{\mathrm{BCOV}}(X)$. Then $\tau_{\mathrm{BCOV}}(X)$ depends only on the isomorphism class of X, while the BCOV torsion does depend on the choice of a Kähler metric on X. Because of this invariance property, the BCOV invariant τ_{BCOV} gives rise to a function on the moduli space of Calabi–Yau threefolds and is identified with the partition function F_1 in [**3**]. In this paper, we give an explicit formula for the BCOV invariant for a class of Calabi–Yau threefolds studied by Borcea [**9**] and Voisin [**36**]. (See [**14**] for some other examples including the quintic mirror threefolds and the FHSV models.) Let us explain our results.

Let S be a $K3$ surface and let $\theta\colon S\to S$ be an anti-symplectic holomorphic involution. Let T be an elliptic curve and let $-1_T\colon T\to T$ be the involution defined as $-1_T(x)=-x$. Let $X_{(S,\theta,T)}$ be the blow-up of the orbifold $(S\times T)/\theta\times(-1)_T$ along the singular locus. Then $X_{(S,\theta,T)}$ is a smooth Calabi–Yau threefold equipped with the following two fibrations. Let $\pi_1\colon X_{(S,\theta,T)}\to S/\theta$ be the elliptic fibration with constant fiber T induced from the projection $(S\times T)/\theta\times(-1)_T\to S/\theta$ and let $\pi_2\colon X_{(S,\theta,T)}\to T/(-1_T)$ be the $K3$-fibration with constant fiber S induced from the projection $(S\times T)/\theta\times(-1)_T\to T/(-1_T)$. The triplet $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is called the Borcea–Voisin threefold associated with (S,θ,T). The moduli space of the triplet $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is determined by the lattice $H^2_-(S,\mathbf{Z})$, the anti-invariant part of the θ-action on $H^2(S,\mathbf{Z})$. By **[28]**, $H^2_-(S,\mathbf{Z})$ is isometric to a primitive 2-elementary sublattice of the $K3$-lattice $\mathbb{L}_{K3}$. Let $\Lambda \subset \mathbb{L}_{K3}$ be a sublattice of rank $r(\Lambda)$. A Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is of type Λ if $H^2_-(S,\mathbf{Z})$ is isometric to Λ. Since θ is anti-symplectic, there exist Borcea–Voisin threefolds of type Λ if and only if $\Lambda \subset \mathbb{L}_{K3}$ is a primitive 2-elementary sublattice of signature $(2, r(\Lambda)-2)$.

Some Borcea–Voisin threefolds are related to Del Pezzo surfaces. Let V be a Del Pezzo surface and set $\deg V = c_1(V)^2 \in \mathbf{Z}_{>0}$. Let $H(V,\mathbf{Z})$ be the total cohomology group of V, which is equipped with the cup-product $\langle\cdot,\cdot\rangle_V$. Then the sublattice $H^2(V,\mathbf{Z}) \subset H(V,\mathbf{Z})$ is Lorentzian. Let $H(V,\mathbf{Z})(2)$ be the lattice $(H(V,\mathbf{Z}), 2\langle\cdot,\cdot\rangle_V)$. By the classification of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ **[29]**, there exist Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$. Let us explain their moduli space briefly.

Let $\mathscr{K}_V \subset H^2(V,\mathbf{R})$ be the Kähler cone of V, let $\mathscr{C}_V^+ \subset H^2(V,\mathbf{R})$ be the component of the positive cone of $H^2(V,\mathbf{R})$ with $\mathscr{K}_V \subset \mathscr{C}_V^+$ and let $\mathrm{Eff}(V) \subset H^2(V,\mathbf{Z})$ be the set of effective classes on V. The tube domain $H^2(V,\mathbf{R}) + i\,\mathscr{C}_V^+$ is isomorphic to a bounded symmetric domain of type IV and its subdomain $H^2(V,\mathbf{R}) + i\,\mathscr{K}_V$ is called the complexified Kähler cone of V. Let $\mathfrak{H}$ be the complex upper half-plane. By assigning $(X_{(S,\theta,T)},\pi_1,\pi_2)$ the periods of (S,θ) and T, the coarse moduli space of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ is isomorphic to the quotient of the tube domain $(H^2(V,\mathbf{R})+i\,\mathscr{C}_V^+)\times\mathfrak{H}$ by the group $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$ with some divisor removed (cf. Theorem 3.7), where $O^+(H(V,\mathbf{Z}))$ is the group of isometries of $H(V,\mathbf{Z})$ preserving $H^2(V,\mathbf{R}) + i\,\mathscr{C}_V^+$. Hence τ_{BCOV} is regarded as an $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$-invariant function on a certain Zariski open subset of $(H^2(V,\mathbf{R})+i\,\mathscr{C}_V^+)\times\mathfrak{H}$. The goal of this paper is to give an explicit formula for τ_{BCOV} as a function on $(H^2(V,\mathbf{R}) + i\,\mathscr{C}_V^+)\times\mathfrak{H}$ for Borcea–Voisin threefolds of type $H^2(V,\mathbf{Z})(2)$. Let us explain the infinite product appearing in the formula.

After Borcherds **[12]** and Gritsenko–Nikulin **[16]**, we introduce the following infinite product $\Phi_V(z)$ on the complexified Kähler cone $H^2(V,\mathbf{R}) + i\,\mathscr{K}_V$:

$$\Phi_V(z) = e^{\pi i\langle c_1(V),z\rangle_V} \prod_{\alpha\in\mathrm{Eff}(V)} \left(1-e^{2\pi i\langle\alpha,z\rangle_V}\right)^{c^{(0)}_{\deg V}(\alpha^2)}$$
$$\times \prod_{\beta\in\mathrm{Eff}(V),\ \beta/2\equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})} \left(1-e^{\pi i\langle\beta,z\rangle_V}\right)^{c^{(1)}_{\deg V}(\beta^2/4)},$$

where $c^{(0)}_k(m)$ and $c^{(1)}_k(m)$ are the m-th Fourier coefficients of the modular forms

$$f^{(0)}_k(\tau) = \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\,\theta_{\mathbb{A}_1^+}(\tau)^k, \quad f^{(1)}_k(\tau) = -8\,\eta(4\tau)^8\eta(2\tau)^{-16}\,\theta_{\mathbb{A}_1^+ + \frac{1}{2}}(\tau)^k,$$

respectively. Here $\eta(\tau)$ is the Dedekind η-function and $\theta_{\mathbb{A}_1^+}(\tau)$, $\theta_{\mathbb{A}_1^+ + 1/2}(\tau)$ are the theta series of the A_1-lattice. Let $A_{H(V,\mathbf{Z})(2)}$ be the discriminant group of the lattice $H(V,\mathbf{Z})(2)$ and let $\{\mathbf{e}_\gamma\}_{\gamma\in A_{H(V,\mathbf{Z})(2)}}$ be the standard basis of $\mathbf{C}[A_{H(V,\mathbf{Z})(2)}]$, the group ring of $A_{H(V,\mathbf{Z})(2)}$. In Sects. 4.3, 4.4 and 6.2, we shall prove that $\Phi_V(2z)^2$ is the Borcherds lift [**12**] of the $\mathbf{C}[A_{H(V,\mathbf{Z})(2)}]$-valued elliptic modular form

$$f^{(0)}_{\deg V}(\tau)\,\mathbf{e}_0 + \sum_{\gamma\in A_{H(V,\mathbf{Z})(2)}}\ \sum_{m\equiv 2\gamma^2 \mod 4} c^{(0)}_{\deg V}(m)\,q^{m/4}\,\mathbf{e}_\gamma + f^{(1)}_{\deg V}(\tau)\,\mathbf{e}_{\mathbf{1}_{H(V,\mathbf{Z})(2)}}$$

with respect to the lattice $H(V,\mathbf{Z})(2)$. Here $\mathbf{1}_{H(V,\mathbf{Z})(2)} \in A_{H(V,\mathbf{Z})(2)}$ is the characteristic element and $q = \exp(2\pi i\tau)$. As a result, $\Phi_V(z)$ converges when $(\operatorname{Im} z)^2 \gg 0$ and extends to an automorphic form on $H^2(V,\mathbf{R}) + i\,\mathscr{C}^+_V$ for $O^+(H(V,\mathbf{Z}))$ of weight $\deg V + 4$ vanishing exactly on the Heegner divisor of norm (-1)-vectors of $H(V,\mathbf{Z})$. If $\operatorname{Exc}(V) \subset H^2(V,\mathbf{Z})$ denotes the exceptional classes on V, the following functional equations hold by the automorphic property of $\Phi_V(z)$ (cf. Sect. 6.3):

(a) $\Phi_V(z+l) = \Phi_V(z)$ for all $l \in H^2(V,\mathbf{Z})$ with $\langle l, c_1(V)\rangle_V \equiv 0 \mod 2$.
(b) $\Phi_V(g(z)) = \pm\Phi_V(z)$ for all $g \in O^+(H^2(V,\mathbf{Z}))$.
(c) $\Phi_V(-\frac{z}{\langle z,z\rangle_V} + \delta) = -(-\langle z,z\rangle_V)^{\deg V+4}\,\Phi_V(z+\delta)$ for all $\delta \in \operatorname{Exc}(V)$.
(d) $\Phi_V(-\frac{2z}{\langle z,z\rangle_V}) = (-\frac{\langle z,z\rangle_V}{2})^{\deg V+4}\,\Phi_V(z)$.

Since $c_1(V)/2$ is a Weyl vector of $H^2(V,\mathbf{Z})(2)$, the Fourier expansion of $\Phi_V(2z)$ is of Lie type in the sense of [**18**] by (a), (b). Hence there exists a Borcherds superalgebra whose denominator function is $\Phi_V(2z)$. This Borcherds superalgebra is obtained as an automorphic correction [**17**] of the Kac-Moody algebra defined by the generalized Cartan matrix $(2\langle c_1(E), c_1(E')\rangle_V)_{E,E'\in\operatorname{Exc}(V)}$. (See Question 4.4.)

Let $\|\Phi_V\|$ and $\|\eta\|$ be the Petersson norms of $\Phi_V(z)$ and $\eta(\tau)$, respectively. Then $\|\Phi_V\|^2 \cdot \|\eta^{24}\|^2$ is a function on $(H^2(V,\mathbf{R}) + i\,\mathscr{C}^+_V) \times \mathfrak{H}$ invariant under the action of $O^+(H(V,\mathbf{Z})) \times SL_2(\mathbf{Z})$. The following (cf. Theorems 5.7 and 6.4) is the main result of this paper.

Theorem 1.1. — *If V is a Del Pezzo surface with $1 \leq \deg V \leq 6$, then there exists a constant $C_{\deg V}$ depending only on $\deg V$ such that the following equation of functions on the moduli space of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ holds:*

$$\tau_{\mathrm{BCOV}} = C_{\deg V}\,\|\Phi_V\|^2 \cdot \|\eta^{24}\|^2.$$

Under the identification of τ_{BCOV} with F_1 in B-model [**2**], [**3**], it follows from Theorem 1.1 that the conjecture of Harvey–Moore [**19**, Sect. 7] holds for Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ when $1 \leq \deg V \leq 6$, since Φ_V is the denominator function of a Borcherds superalgebra.

After Theorem 1.1, the conjecture of Bershadsky–Cecotti–Ooguri–Vafa [**2**], [**3**] seems to predict that the elliptic Gromov–Witten invariants of the mirror of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ are expressed as certain linear combinations of the Fourier coefficients $c^{(0)}_{\deg V}(m)$, $c^{(1)}_{\deg V}(m)$. If this is the case, the invariant of $K3$

surfaces with involution constructed in [**37**] would be the Borcherds lift of an elliptic modular form whose Fourier coefficients are elliptic Gromov–Witten invariants of some Calabi–Yau threefolds by the structure theorem [**38**, Th. 0.1]. However, since the Borcea–Voisin construction of mirrors [**9**], [**36**] does not apply to Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$, we do not know the existence of mirrors for those Borcea–Voisin threefolds as well as their elliptic Gromov–Witten invariants.

This paper is organized as follows. In Sect. 2, we recall some definitions and results about lattices. In Sect. 3, we recall Borcea–Voisin threefolds and study their moduli space. In Sect. 4, we introduce the automorphic form Φ_m, which will be identified with $\mathit{\Phi}_V$ in Sect. 6. In Sect. 5, we recall the BCOV invariant of a Calabi–Yau threefold and we prove the main theorem. In Sect. 6, we rewrite the automorphic form Φ_m as an automorphic form on the complexified Kähler cone of a Del Pezzo surface to give an identification between Φ_m and $\mathit{\Phi}_V$.

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2. Lattices and orthogonal modular varieties

A free $\mathbf{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form is called a lattice. We often identify a non-degenerate, integral, symmetric matrix with the corresponding lattice. The rank of a lattice L is denoted by $r(L)$. The signature of L is denoted by $\operatorname{sign}(L) = (b^+(L), b^-(L))$. A lattice L is *Lorentzian* if $\operatorname{sign}(L) = (1, r(L)-1)$. For a lattice L with bilinear form $\langle\cdot,\cdot\rangle$, we denote by $L(k)$ the lattice with bilinear form $k\langle\cdot,\cdot\rangle$. The set of roots of L is defined by $\Delta_L := \{d \in L;\ \langle d,d\rangle = -2\}$. The isometry group of L is denoted by $O(L)$. For $r \in L\otimes\mathbf{R}$, the reflection $s_r \in O(L\otimes\mathbf{R})$ is defined by $s_r(x) = x - 2\frac{\langle x,r\rangle}{\langle r,r\rangle} r$ for $x \in L\otimes\mathbf{R}$. If $\delta \in L$ and $\delta^2 = -1$ or $\delta^2 = -2$, then $s_\delta \in O(L)$. The subgroup of $O(L)$ generated by the reflections $\{s_\delta\}_{\delta\in\Delta_L}$ is called the *Weyl group* of L and is denoted by $W(L)$. The dual lattice of L is defined by $L^\vee := \operatorname{Hom}_{\mathbf{Z}}(L,\mathbf{Z}) \subset L\otimes\mathbf{Q}$. We set $A_L := L^\vee/L$. A lattice L is *unimodular* if $A_L = 0$. A lattice L is *even* if $\langle x,x\rangle \in 2\mathbf{Z}$ for all $x \in L$. A lattice is *odd* if it is not even. A sublattice $M \subset L$ is *primitive* if L/M has no torsion elements.

2.1. 2-elementary lattices. — Set $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. An even lattice L is 2-*elementary* if there is an integer $l \geq 0$ with $A_L \cong \mathbf{Z}_2^l$. For a 2-elementary lattice L, we set $l(L) := \dim_{\mathbf{Z}_2} A_L$.

Let $\mathbb{U} = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and let $\mathbb{A}_1$, $\mathbb{E}_8$ be the *negative-definite* Cartan matrix of type A_1, E_8 respectively, which are identified with the corresponding even lattices. Then $\mathbb{U}$ and